科目名	解析学	対象	2OB	学部研究科	理学部第一部	学科専攻科		学籍番号		評点
平成 25 年 1 月 17 日 (木) 2 回目 (〜 時限目)		担当	石川 学	学年		氏名				
試験時間	60 分	注意事項	① 筆記用具以外持込不可 ~~2. 下記のみ参照・持込可~~ ( )							

平成 24 年度後期定期試験

※解答用紙の裏面使用可

1 次の関数の $z=0$ を中心とする Laurent（ローラン）展開を与えられた範囲で求めよ．

(1) $\dfrac{1}{z^3(z-4)^2}$ $(0<|z|<4)$　　(2) $\dfrac{7z}{(z-4)(z+3)}$ $(3<|z|<4)$

2 次の場合に，留数 $\mathrm{Res}[f,a]$ を求めよ．

(1) $f(z)=\dfrac{2z-1}{(z+4)(z-1)^2}$, $a=-4$　　(2) $f(z)=\dfrac{\sin(i\pi z)}{z(6z-i)(z+i)}$, $a=\dfrac{i}{6}$

(3) $f(z)=\dfrac{z^3-2}{z(z+2)^3}$, $a=-2$　　(4) $f(z)=z^3\cos\dfrac{3}{z}$, $a=0$

3 次の積分を求めよ．ただし，積分経路は正の向きとする．

(1) $\displaystyle\int_{|z-5|=2}\frac{3z+4}{(z-5)(z+1)}dz$　　(2) $\displaystyle\int_{|z|=7}\frac{z+1}{(z+3)(z+5)}dz$

(3) $\displaystyle\int_{|z-1|=2}\frac{1}{z(z-1)(z+2)}dz$　　(4) $\displaystyle\int_{|z|=2}\frac{1}{(z^2+1)(z+3)}dz$

(5) $\displaystyle\int_{|z|=1}\frac{z^4+1}{(2z+1)(z-2)}dz$　　(6) $\displaystyle\int_{|z|=2}\frac{z+1}{z(2z-3)(z+4)}dz$

(7) $\displaystyle\int_{|z|=1}\frac{\cosh z}{z^2(2z+3)}dz$　　(8) $\displaystyle\int_{|z|=2}\frac{1}{z^3(z-1)(z+3)}dz$

4 次の積分を求めよ．

(1) $\displaystyle\int_0^{2\pi}\frac{1}{13-12\cos\theta}d\theta$　　(2) $\displaystyle\int_{-\infty}^{\infty}\frac{2x^2-1}{x^4+6x^2+25}dx$

東京理科大学　平成　　年　　月　　日　**試験答案**

科目名		担当	先生	評点
該当学部を○で囲む　理学部 1 / 工学部 2	部　学科　年　番	氏名	H24 後・定	

[1]

(1) $0<|z|<4$ のとき $\left|\frac{z}{4}\right|<1$ だから

$$\frac{1}{z-4} = -\frac{1}{4}\cdot\frac{1}{1-\frac{z}{4}} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}}$$

微分して

$$\cancel{-}\frac{1}{(z-4)^2} = \cancel{-}\sum_{n=0}^{\infty}\frac{nz^{n-1}}{4^{n+1}}$$

$$\therefore \frac{1}{z^3(z-4)^2} = \sum_{n=0}^{\infty}\frac{nz^{n-4}}{4^{n+1}} \quad (0<|z|<4)$$

(2) $\dfrac{7z}{(z-4)(z+3)} = \dfrac{4}{z-4} + \dfrac{3}{z+3}$ である

$3<|z|$ のとき $\left|\frac{3}{z}\right|<1$ だから

$$\frac{3}{z+3} = \frac{3}{z}\cdot\frac{1}{1-\left(-\frac{3}{z}\right)} = \frac{3}{z}\sum_{n=0}^{\infty}\left(-\frac{3}{z}\right)^n = \sum_{n=0}^{\infty}\frac{-(-3)^{n+1}}{z^{n+1}}$$

$$\therefore \frac{7z}{(z-4)(z+3)} = -\sum_{n=0}^{\infty}\frac{z^n}{4^n} - \sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{z^{n+1}} \quad (3<|z|<4)$$

[2]

(1) $\mathrm{Res}[f,a] = \lim_{z\to -4}(z+4)f(z) = \lim_{z\to -4}\dfrac{2z-1}{(z-1)^2} = -\dfrac{9}{25}$

(2) $\mathrm{Res}[f,a] = \lim_{z\to \frac{i}{6}}\left(z-\frac{i}{6}\right)f(z) = \lim_{z\to\frac{i}{6}}\dfrac{\sin(i\pi z)}{6z(z+i)} = \dfrac{3}{7}$

(3) $\mathrm{Res}[f,a] = \dfrac{1}{2!}\lim_{z\to -2}\left\{(z+2)^3 f(z)\right\}^{(2)} = \dfrac{1}{2}\lim_{z\to -2}\left(\dfrac{z^3-2}{z}\right)''$

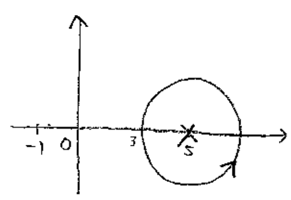
$$= \frac{1}{2}\lim_{z\to -2}\left(z^2-2z^{-1}\right)'' = \frac{1}{2}\lim_{z\to -2}\left(2-4z^{-3}\right) = \frac{5}{4}$$

(4) $z^3\cos\frac{3}{z} = z^3\sum_{n=0}^{\infty}\dfrac{(-1)^n}{(2n)!}\left(\dfrac{3}{z}\right)^{2n} = z^3 - \dfrac{9}{2}z + \dfrac{27}{8z} - \cdots$

$$\therefore \mathrm{Res}[f,a] = \frac{27}{8}$$

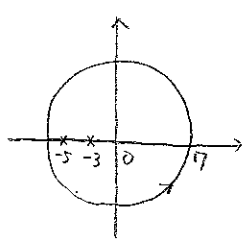
[3]

(1) $\displaystyle\int_{|z-5|=2}\frac{3z+4}{(z-5)(z+1)}\,dz$



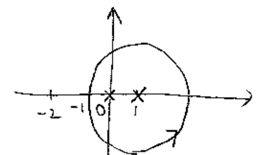
$$= 2\pi i\cdot\left.\frac{3z+4}{z+1}\right|_{z=5} = \frac{19}{3}\pi i$$

(2) $\displaystyle\int_{|z|=7}\frac{z+1}{(z+3)(z+5)}\,dz$



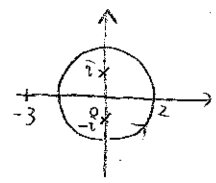
$$= 2\pi i\left(\left.\frac{z+1}{z+5}\right|_{z=-3} + \left.\frac{z+1}{z+3}\right|_{z=-5}\right) = 2\pi i$$

(3) $\displaystyle\int_{|z-1|=2}\frac{1}{z(z-1)(z+2)}\,dz$



$$= 2\pi i\left\{\left.\frac{1}{(z-1)(z+2)}\right|_{z=0} + \left.\frac{1}{z(z+2)}\right|_{z=1}\right\} = -\frac{\pi}{3}i$$

(4) $\displaystyle\int_{|z|=2}\frac{1}{(z^2+1)(z+3)}\,dz$

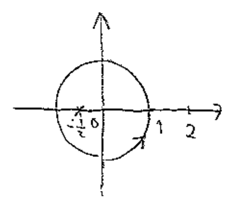


$$= 2\pi i\left\{\left.\frac{1}{(z+i)(z+3)}\right|_{z=i} + \left.\frac{1}{(z-i)(z+3)}\right|_{z=-i}\right\}$$

$$= 2\pi i\left\{\frac{1}{2i(i+3)} + \frac{1}{-2i(-i+3)}\right\}$$

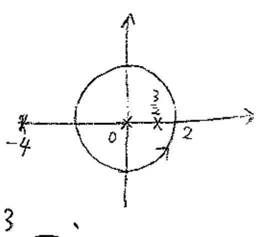
$$= \pi\left(\frac{1}{3+i} - \frac{1}{3-i}\right) = \frac{-2i\pi}{9+1} = -\frac{\pi}{5}i$$

(5) $\displaystyle\int_{|z|=1}\frac{z^4+1}{(2z+1)(z-2)}\,dz$



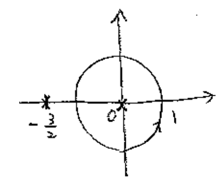
$$= 2\pi i\cdot\left.\frac{z^4+1}{2(z-2)}\right|_{z=-\frac{1}{2}} = -\frac{17}{40}\pi i$$

(6) $\displaystyle\int_{|z|=2}\frac{z+1}{z(2z-3)(z+4)}\,dz$



$$= 2\pi i\left\{\left.\frac{z+1}{(2z-3)(z+4)}\right|_{z=0} + \left.\frac{z+1}{2z(z+4)}\right|_{z=\frac{3}{2}}\right\} = \frac{3}{22}\pi i$$

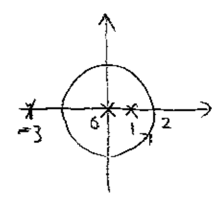
(7) $\displaystyle\int_{|z|=1}\frac{\cosh z}{z^2(2z+3)}\,dz$



$$= \frac{2\pi i}{1!}\cdot\left.\left(\frac{\cosh z}{2z+3}\right)'\right|_{z=0}$$

$$= 2\pi i\cdot\left.\frac{\sinh z\cdot(2z+3) - \cosh z\cdot 2}{(2z+3)^2}\right|_{z=0} = -\frac{4}{9}\pi i$$

(8) $\displaystyle\int_{|z|=2}\frac{1}{z^3(z-1)(z+3)}\,dz$



$$= 2\pi i\cdot\left.\frac{1}{z^3(z+3)}\right|_{z=1} + \frac{2\pi i}{2!}\cdot\left.\left\{\frac{1}{(z-1)(z+3)}\right\}''\right|_{z=0}$$

$$= 2\pi i\cdot\frac{1}{4} + \pi i\cdot\frac{1}{4}\left.\left\{(z-1)^{-1}-(z+3)^{-1}\right\}''\right|_{z=0}$$

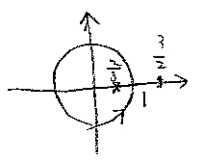
$$= \frac{\pi}{2}i + \frac{\pi}{4}i\left.\left\{2(z-1)^{-3} - 2(z+3)^{-3}\right\}\right|_{z=0}$$

$$= \cancel{\frac{\pi}{2}i} + \frac{\pi}{2}i\cdot\left(\cancel{-1} - \frac{1}{27}\right) = -\frac{\pi}{54}i$$

$\boxed{4}$

(1) $$\int_0^{2\pi}\frac{1}{13-12\cos\theta}\,d\theta=\int_{|z|=1}\frac{1}{13-12\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{1}{iz}\,dz$$

$$=\frac{-1}{i}\int_{|z|=1}\frac{1}{6z^2-13z+6}\,dz$$



$$=\frac{-1}{i}\int_{|z|=1}\frac{1}{(3z-2)(2z-3)}\,dz$$

$$=\frac{-1}{i}\cdot 2\pi i\cdot\left.\frac{1}{3(2z-3)}\right|_{z=\frac{2}{3}}$$

$$=\frac{2}{5}\pi$$

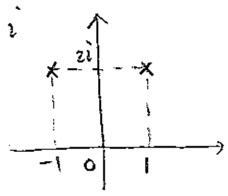
(2) $$x^4+6x^2+25=(x^2+5)^2-4x^2$$
$$=(x^2+2x+5)(x^2-2x+5)$$

より，$x^4+6x^2+25=0$ の解は

$$x=-1\pm 2i,\ 1\pm 2i$$

上半平面にあるのは $x=-1+2i,\ 1+2i$

分母：4次，分子：2次 だから



$$\int_{-\infty}^{\infty}\frac{2x^2-1}{x^4+6x^2+25}\,dx$$

$$=2\pi i\left(\mathrm{Res}\left[\frac{2z^2-1}{z^4+6z^2+25},\,-1+2i\right]+\mathrm{Res}\left[\frac{2z^2-1}{z^4+6z^2+25},\,1+2i\right]\right)$$

$$=2\pi i\left(\left.\frac{2z^2-1}{4z^3+12z}\right|_{z=\underbrace{-1+2i}_{=\alpha}}+\left.\frac{2z^2-1}{4z^3+12z}\right|_{z=\underbrace{1+2i}_{=\beta}}\right)$$

$$=2\pi i\left\{\frac{2\alpha^2-1}{4\alpha(\alpha^2+3)}+\frac{2\beta^2-1}{4\beta(\beta^2+3)}\right\}$$

$$\left(\begin{array}{l}\alpha^2+3=(1-4i-4)+3=-4i\\ \beta^2+3=(1+4i-4)+3=4i\end{array}\right)$$

$$=2\pi i\left\{\frac{2\alpha^2-1}{4\alpha\cdot(-4i)}+\frac{2\beta^2-1}{4\beta\cdot 4i}\right\}$$

$$=\frac{\pi}{8}\left(\frac{2\alpha^2-1}{-\alpha}+\frac{2\beta^2-1}{\beta}\right)$$

$$=\frac{\pi}{8}\left(-2\alpha+\frac{1}{\alpha}+2\beta-\frac{1}{\beta}\right)$$

$$=\frac{\pi}{8}\left\{2(\beta-\alpha)+\frac{\beta-\alpha}{\alpha\beta}\right\}$$

$$=\frac{\pi}{8}\left(2+\frac{1}{\alpha\beta}\right)(\beta-\alpha)$$

$$=\frac{\pi}{8}\left(2+\frac{1}{-1-4}\right)\cdot 2$$

$$=\frac{9}{20}\pi$$