科目名	解析学	対象	2OB	学部 研究科	理学部第一部	学科 専攻科		学籍 番号	
平成 26 年 1 月 23 日 (木) 3 回目 (~ 時限目)				担当	石川 学	学年		氏名	
試験 時間	60 分	注意 事項	1. 筆記用具以外持込不可 ~~2. 下記のみ参照・持込可~~ ( )						

評点

平成 25 年度後期定期試験

※解答用紙の裏面使用可

$\boxed{1}$ 次の関数の $z=0$ を中心とする Laurent (ローラン) 展開を与えられた範囲で求めよ. (10 点)

(1) $\dfrac{1}{z^4(z-5)^2}$ $(0<|z|<5)$ (2) $\dfrac{7z-5}{(z-5)(z+1)}$ $(1<|z|<5)$

$\boxed{2}$ 次の場合に，留数 $\mathrm{Res}[f,a]$ を求めよ. (25 点)

(1) $f(z)=\dfrac{2z-3}{(z+4)(z+2)^3}$, $a=-4$ (2) $f(z)=\dfrac{\cos(i\pi z)}{z(3z-i)(z+i)}$, $a=\dfrac{i}{3}$

(3) $f(z)=\dfrac{z^3-2z+1}{z(z+3)^3}$, $a=-3$ (4) $f(z)=z^4\sin\dfrac{3}{z}$, $a=0$

(5) $f(z)=\dfrac{z}{\sin z}$, $a=n\pi$ （ただし n は自然数とする）

$\boxed{3}$ 次の積分を求めよ．ただし，積分経路は正の向きとする. (40 点)

(1) $\displaystyle\int_{|z-5|=2}\frac{3z+2}{(z-5)(z+1)}dz$ (2) $\displaystyle\int_{|z|=7}\frac{z^2}{(z+3)(z-5)}dz$

(3) $\displaystyle\int_{|z+1|=2}\frac{1}{z(z+1)(z-2)}dz$ (4) $\displaystyle\int_{|z|=3}\frac{z}{(z^2+4)(z-7)}dz$

(5) $\displaystyle\int_{|z|=1}\frac{z^2-1}{(4z+1)(z-2)}dz$ (6) $\displaystyle\int_{|z|=2}\frac{3z-2}{z(2z-3)(z+4)}dz$

(7) $\displaystyle\int_{|z|=1}\frac{\cosh z}{z^2(z+2)}dz$ (8) $\displaystyle\int_{|z|=2}\frac{2}{z^4(z+1)^2(z+3)}dz$

$\boxed{4}$ 次の積分を求めよ. (25 点)

(1) $\displaystyle\int_0^{2\pi}\frac{1}{25-24\cos\theta}d\theta$ (2) $\displaystyle\int_{-\infty}^{\infty}\frac{5x-7}{(x^2+9)(x^2-4x+13)}dx$

東京理科大学　　　　年　　月　　日 **試　験　答　案**

科目名		担当	先生
所属 学籍番号	学部　学科　番	氏名	H25 後・定

評点

$\boxed{1}$

(1) $0<|z|<5$ のとき $\left|\frac{z}{5}\right|<1$ だから

$$\frac{1}{z-5}=-\frac{1}{5}\cdot\frac{1}{1-\frac{z}{5}}=-\frac{1}{5}\sum_{m=0}^{\infty}\left(\frac{z}{5}\right)^m=-\sum_{m=0}^{\infty}\frac{z^m}{5^{m+1}}$$

ビブンして

$$-\frac{1}{(z-5)^2}=-\sum_{m=0}^{\infty}\frac{mz^{m-1}}{5^{m+1}}\qquad\therefore\ \frac{1}{(z-5)^2}=\sum_{m=0}^{\infty}\frac{mz^{m-1}}{5^{m+1}}$$

$$\therefore\ \frac{1}{z^4(z-5)^2}=\sum_{m=0}^{\infty}\frac{mz^{m-5}}{5^{m+1}}\qquad(0<|z|<5)$$

(2) $\frac{7z-5}{(z-5)(z+1)}=\frac{5}{z-5}+\frac{2}{z+1}$ である．

$1<|z|<5$ のとき，(1) より

$$\frac{5}{z-5}=-\sum_{m=0}^{\infty}\frac{z^m}{5^m}$$

また，$\left|\frac{1}{z}\right|<1$ だから

$$\frac{2}{z+1}=\frac{2}{z}\cdot\frac{1}{1-\left(-\frac{1}{z}\right)}=\frac{2}{z}\sum_{m=0}^{\infty}\left(-\frac{1}{z}\right)^m=\sum_{m=0}^{\infty}\frac{2\cdot(-1)^m}{z^{m+1}}$$

$$\therefore\ \frac{7z-5}{(z-5)(z+1)}=-\sum_{m=0}^{\infty}\frac{z^m}{5^m}+\sum_{m=0}^{\infty}\frac{2\cdot(-1)^m}{z^{m+1}}\qquad(1<|z|<5)$$

$\boxed{2}$

(1) $\mathrm{Res}[f,a]=\lim_{z\to-4}(z+4)f(z)=\lim_{z\to-4}\frac{2z-3}{(z+2)^3}=\frac{11}{8}$

(2) $\mathrm{Res}[f,a]=\lim_{z\to\frac{i}{3}}\left(z-\frac{i}{3}\right)f(z)=\lim_{z\to\frac{i}{3}}\frac{\cos(i\pi z)}{3z(z+i)}=-\frac{3}{8}$

(3) $\mathrm{Res}[f,a]=\frac{1}{2!}\lim_{z\to-3}\{(z+3)^3f(z)\}^{(2)}=\frac{1}{2}\lim_{z\to-3}\left(\frac{z^3-2z+1}{z}\right)''$

$$=\frac{1}{2}\lim_{z\to-3}\left(z^2-2+z^{-1}\right)''=\frac{1}{2}\lim_{z\to-3}\left(2+2z^{-3}\right)=\frac{26}{27}$$

(4) $z\neq0$ のとき

$$\sin\frac{3}{z}=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left(\frac{3}{z}\right)^{2m+1}=\frac{3}{z}-\frac{1}{6}\cdot\frac{3^3}{z^3}+\frac{1}{120}\cdot\frac{3^5}{z^5}-\cdots$$

$$=\frac{3}{z}-\frac{9}{2z^3}+\frac{81}{40z^5}-\cdots$$

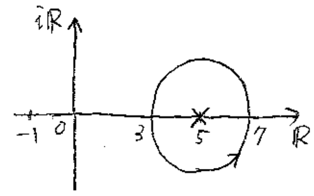
$$\therefore\ z^4\sin\frac{3}{z}=3z^3-\frac{9}{2}z+\frac{81}{40z}-\cdots$$

$$\therefore\ \mathrm{Res}[f,a]=\frac{81}{40}$$

(5) $\mathrm{Res}[f,a]=\left.\frac{z}{(\sin z)'}\right|_{z=m\pi}=\left.\frac{z}{\cos z}\right|_{z=n\pi}=(-1)^m m\pi$

$\boxed{3}$

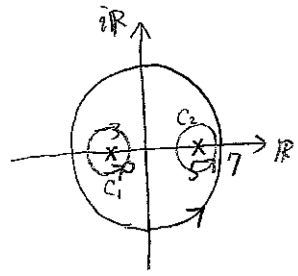
(1) $\displaystyle\int_{|z-5|=2}\frac{3z+2}{(z-5)(z+1)}dz$



$$=\int_{|z-5|=2}\frac{\frac{3z+2}{z+1}}{z-5}dz$$

$$=2\pi i\cdot\left.\frac{3z+2}{z+1}\right|_{z=5}=\frac{17}{3}\pi i$$

(2) $\displaystyle\int_{|z|=7}\frac{z^2}{(z+3)(z-5)}dz$

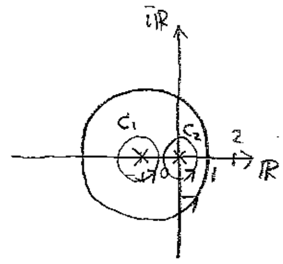


$$=\int_{C_1}\frac{\frac{z^2}{z-5}}{z+3}dz+\int_{C_2}\frac{\frac{z^2}{z+3}}{z-5}dz$$

$$=2\pi i\cdot\left(\left.\frac{z^2}{z-5}\right|_{z=-3}+\left.\frac{z^2}{z+3}\right|_{z=5}\right)$$

$$=2\pi i\left(-\frac{9}{8}+\frac{25}{8}\right)=4\pi i$$

(3) $\displaystyle\int_{|z+1|=2}\frac{1}{z(z+1)(z-2)}dz$

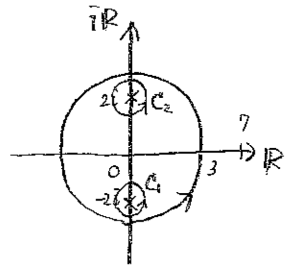


$$=\int_{C_1}\frac{\frac{1}{z(z-2)}}{z+1}dz+\int_{C_2}\frac{\frac{1}{(z+1)(z-2)}}{z}dz$$

$$=2\pi i\left\{\left.\frac{1}{z(z-2)}\right|_{z=-1}+\left.\frac{1}{(z+1)(z-2)}\right|_{z=0}\right\}$$

$$=2\pi i\left(\frac{1}{3}-\frac{1}{2}\right)=-\frac{\pi}{3}i$$

(4) $\displaystyle\int_{|z|=3}\frac{z}{(z^2+4)(z-7)}dz$

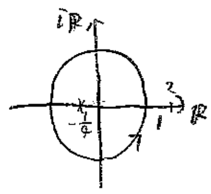


$$=\int_{C_1}\frac{\frac{z}{(z-2i)(z-7)}}{z+2i}dz+\int_{C_2}\frac{\frac{z}{(z+2i)(z-7)}}{z-2i}dz$$

$$=2\pi i\left\{\left.\frac{z}{(z-2i)(z-7)}\right|_{z=-2i}+\left.\frac{z}{(z+2i)(z-7)}\right|_{z=2i}\right\}$$

$$=\cancel{2}\pi i\left\{\frac{1}{\cancel{2}(-2i-7)}+\frac{1}{\cancel{2}(2i-7)}\right\}=\pi i\cdot\frac{\cancel{2i}-7-\cancel{2i}-7}{4+49}=-\frac{14}{53}\pi i$$

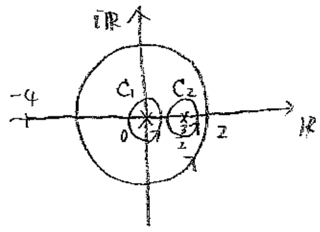
(5) $\displaystyle\int_{|z|=1}\frac{z^2-1}{(4z+1)(z-2)}dz$



$$=\int_{|z|=1}\frac{\frac{z^2-1}{4(z-2)}}{z+\frac{1}{4}}dz$$

$$=2\pi i\cdot\left.\frac{z^2-1}{4(z-2)}\right|_{z=-\frac{1}{4}}=\frac{5}{24}\pi i$$

(6) $\displaystyle\int_{|z|=2} \frac{3z-2}{z(2z-3)(z+4)}\,dz$

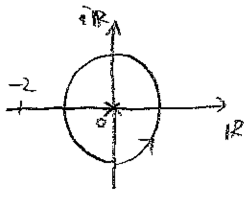


$\displaystyle = \int_{C_1} \frac{\frac{3z-2}{(2z-3)(z+4)}}{z}\,dz + \int_{C_2} \frac{\frac{3z-2}{2z(z+4)}}{z-\frac{3}{2}}\,dz$

$\displaystyle = 2\pi i \left\{ \frac{3z-2}{(2z-3)(z+4)}\Big|_{z=0} + \frac{3z-2}{2z(z+4)}\Big|_{z=\frac{3}{2}} \right\}$

$\displaystyle = 2\pi i \left(\frac{1}{6} + \frac{5}{33} \right) = \frac{7}{11}\pi i$

(7) $\displaystyle\int_{|z|=1} \frac{\cosh z}{z^2(z+2)}\,dz$

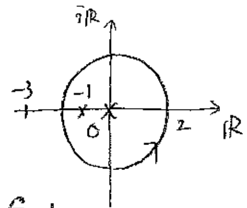


$\displaystyle = \int_{|z|=1} \frac{\frac{\cosh z}{z+2}}{z^2}\,dz$

$\displaystyle = \frac{2\pi i}{1!}\left(\frac{\cosh z}{z+2}\right)'\Big|_{z=0} = 2\pi i \cdot \frac{\sinh z\cdot(z+2) - \cosh z\cdot 1}{(z+2)^2}\Big|_{z=0}$

$\displaystyle = 2\pi i \cdot \frac{-1}{4} = -\frac{\pi}{2} i$

(8) $\displaystyle\int_{|z|=2} \frac{2}{z^4(z+1)^2(z+3)}\,dz$



$\displaystyle = \int_{|z|=2} \left\{ \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z^3} + \frac{D}{z^4} + \frac{E}{z+1} + \frac{F}{(z+1)^2} + \frac{G}{z+3} \right\} dz$

$= 2\pi i (A+E)$

である．分解の分母をはらうと

$2 = Az^3(z+1)^2(z+3) + Bz^2(z+1)^2(z+3) + Cz(z+1)^2(z+3) + D(z+1)^2(z+3)$
$\quad + Ez^4(z+1)(z+3) + Fz^4(z+3) + Gz^4(z+1)^2$

z^6 の係数を比較して　$0 = A + E + G$

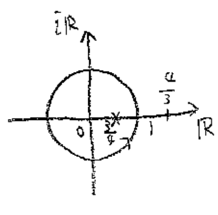
$z=-3$ 代入して　$2 = G\cdot 81\cdot 4$　$\therefore G = +\frac{1}{2\cdot 81}$

$\displaystyle \therefore \int_{|z|=2} \frac{2}{z^4(z+1)^2(z+3)}\,dz = 2\pi i \cdot \frac{-1}{2\cdot 81} = -\frac{\pi}{81} i$

$\boxed{4}$

(1) $\displaystyle\int_0^{2\pi} \frac{1}{25-24\cos\theta}\,d\theta = \int_{|z|=1} \frac{1}{25-24\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)}\cdot\frac{1}{iz}\,dz$

$\displaystyle = \frac{-1}{i}\int_{|z|=1} \frac{1}{12z^2-25z+12}\,dz$



$\displaystyle = \frac{-1}{i}\int_{|z|=1} \frac{1}{(4z-3)(3z-4)}\,dz$

$\displaystyle = \frac{-1}{i}\int_{|z|=1} \frac{\frac{1}{4(3z-4)}}{z-\frac{3}{4}}\,dz$

$\displaystyle = \frac{-1}{i}\cdot 2\pi i \cdot \frac{1}{4(3z-4)}\Big|_{z=\frac{3}{4}}$

$\displaystyle = \frac{2}{7}\pi$

(2) $(z^2+9)(z^2-4z+13)=0$ の解は

$z = \pm 3i,\ 2\pm 3i$

で，このうち上半平面にあるのは

$z = 3i,\ 2+3i$

また

$\left((z^2+9)(z^2-4z+13)\text{の次数}\right) - \left(5z-7\text{の次数}\right) = 3 > 2$

だから

$\displaystyle \int_{-\infty}^{\infty} \frac{5x-7}{(x^2+9)(x^2-4x+13)}\,dx$

$\displaystyle = 2\pi i \left\{ \mathrm{Res}\left(\frac{5z-7}{(z^2+9)(z^2-4z+13)},\ 3i\right) + \mathrm{Res}\left(\frac{5z-7}{(z^2+9)(z^2-4z+13)},\ 2+3i\right) \right\}$

$\displaystyle = 2\pi i \left\{ \frac{5z-7}{(z+3i)(z^2-4z+13)}\Big|_{z=3i} + \frac{5z-7}{(z^2+9)(z-2+3i)}\Big|_{z=2+3i} \right\}$

$\displaystyle = 2\pi i \left\{ \frac{15i-7}{6i(4-12i)} + \frac{15i+3}{(4+12i)6i} \right\}$

$\displaystyle = \frac{2\pi i}{24i}\left(\frac{15i-7}{1-3i} + \frac{15i+3}{1+3i} \right)$

$\displaystyle = \frac{\pi}{12}\cdot\frac{(15i-7)(1+3i)+(15i+3)(1-3i)}{(1-3i)(1+3i)}$

$\displaystyle = \frac{\pi}{12}\cdot\frac{15i-45-7-21i+15i+45+3-9i}{1+9}$

$\displaystyle = \frac{\pi}{12}\cdot\frac{-4}{10}$

$\displaystyle = -\frac{\pi}{30}$