科目名	微積分学	対象	1OB-AB	学部研究科	理学部第一部	学科専攻科		学籍番号		評点
平成 27 年 8 月 3 日 (月) 2 回目 (〜 時限目)				担当	石川 学	学年		氏名		
試験時間	60 分	注意事項	① 筆記用具以外持込不可 ~~2. 下記のみ参照・持込可~~ ( )							

平成 27 年度前期定期試験

※解答用紙の裏面使用可

$\boxed{1}$ 次を求めよ．

(1) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{x}{\tan^3 x}\right)$　(2) $\displaystyle\lim_{x\to 0}(\cos 2x)^{\frac{1}{x\sin \pi x}}$

(3) $\displaystyle\lim_{x\to 0}\frac{x\sqrt{\cos x}-\sin x}{x^3}$　(4) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{0}\arcsin x\,dx$

(5) $\displaystyle\int_{1-\sqrt{2}}^{1}\arctan x\,dx$　(6) $\displaystyle\int_{1}^{\sqrt{3}}\frac{1}{(1+x^2)(\arctan x)^3}dx$

(7) $\displaystyle\int_{\frac{\sqrt{6}-\sqrt{2}}{4}}^{\frac{1}{\sqrt{2}}}\frac{\arccos x}{\sqrt{1-x^2}}dx$　(8) $\displaystyle\int \log(1+x^2)dx$

(9) $\displaystyle\int_{0}^{\frac{1}{2}}\frac{1-x^3}{\sqrt{1-x^2}}dx$　(10) $\displaystyle\int \frac{2x\arctan x}{(1+x^2)^2}dx$

$\boxed{2}$ 次の関数の Maclaurin 展開ををかっこ内の項まで求めよ．必要ならば，下の Maclaurin 展開を用いてよい．ただし，**係数は既約分数**にすること．

(1) $\sqrt{1+x-\dfrac{x^2}{4}}$　(4 次以下)　(2) $e^{x\cos x}$　(5 次以下)

★代表的な関数の Maclaurin 展開

$\sqrt{1+x}=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{16}x^3-\dfrac{5}{128}x^4+\cdots\quad(-1\leqq x\leqq 1)$

$e^x=1+x+\dfrac{1}{2}x^2+\dfrac{1}{6}x^3+\dfrac{1}{24}x^4+\dfrac{1}{120}x^5+\cdots\quad(x\in\mathbb{R})$

$\cos x=1-\dfrac{1}{2}x^2+\dfrac{1}{24}x^4-\cdots\quad(x\in\mathbb{R})$

東京理科大学　　年　月　日　**試験答案**　　評点

科目名		担当	先生
所属 学籍番号	学部　学科　番	氏名	

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(1) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{x}{\tan^3 x}\right)=\lim_{x\to 0}\frac{\tan^3 x-x^3}{x^2\tan^3 x}=\lim_{x\to 0}\frac{(\tan x-x)(\tan^2 x+x\tan x+x^2)}{x^2\tan^3 x}$

$\displaystyle=\lim_{x\to 0}\frac{\tan x-x}{x^3}\cdot\left\{\frac{x}{\tan x}+\left(\frac{x}{\tan x}\right)^2+\left(\frac{x}{\tan x}\right)^3\right\}=\frac{1}{3}\cdot(1+1^2+1^3)=1$

※ $\displaystyle\lim_{x\to 0}\frac{\tan x-x}{x^3}\ \text{Ⓛ}\ \lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-1}{3x^2}=\lim_{x\to 0}\frac{1}{3}\left(\frac{\tan x}{x}\right)^2=\frac{1}{3}\cdot 1^2=\frac{1}{3}$ (これは公式としてよいことにしてあった)

(2) $x\fallingdotseq 0$ のとき $\cos 2x>0$ だから $\displaystyle(\cos 2x)^{\frac{1}{x\sin\pi x}}=e^{\log(\cos 2x)^{\frac{1}{x\sin\pi x}}}=e^{\frac{\log(\cos 2x)}{x\sin\pi x}}$

ここで

$\displaystyle\lim_{x\to 0}\frac{\log(\cos 2x)}{x\sin\pi x}\ \text{Ⓛ}\ \lim_{x\to 0}\frac{\frac{-2\sin 2x}{\cos 2x}}{1\cdot\sin\pi x+x\cdot\pi\cos\pi x}=\lim_{x\to 0}\frac{-2\tan 2x}{\sin\pi x+\pi x\cos\pi x}=\lim_{x\to 0}\frac{-4\cdot\frac{\tan 2x}{2x}}{\frac{\sin\pi x}{\pi x}\cdot\pi+\pi\cos\pi x}=\frac{-4\cdot 1}{1\cdot\pi+\pi\cdot 1}=-\frac{2}{\pi}$

だから (指数関数の連続性より)

$\displaystyle\lim_{x\to 0}(\cos 2x)^{\frac{1}{x\sin\pi x}}=\lim_{x\to 0}e^{\frac{\log(\cos 2x)}{x\sin\pi x}}=e^{-\frac{2}{\pi}}$

(3) $\displaystyle\lim_{x\to 0}\frac{x\sqrt{\cos x}-\sin x}{x^3}=\lim_{x\to 0}\frac{(x\sqrt{\cos x}-x)+(x-\sin x)}{x^3}=\lim_{x\to 0}\left(\frac{\sqrt{\cos x}-1}{x^2}+\frac{x-\sin x}{x^3}\right)=\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}-\frac{1-\sqrt{\cos x}}{x^2}\right)$

$\displaystyle=\lim_{x\to 0}\left(\frac{x-\sin x}{x^3}-\frac{1-\cos x}{x^2}\cdot\frac{1}{1+\sqrt{\cos x}}\right)=\frac{1}{6}-\frac{1}{2}\cdot\frac{1}{1+1}=-\frac{1}{12}$

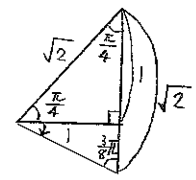
※ $\displaystyle\lim_{x\to 0}\frac{x-\sin x}{x^3}\ \text{Ⓛ}\ \lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{1}{3}\cdot\frac{1-\cos x}{x^2}=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$ (これは公式としてよいことにしてあった)

また, L'Hospital の定理を複数回用いて直接求めてもよい。

(4) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{0}\arcsin x\,dx=\left[x\arcsin x+\sqrt{1-x^2}\right]_{-\frac{\sqrt{3}}{2}}^{0}=\left\{0\cdot 0-\left(-\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{\pi}{3}\right)\right\}+\left(1-\frac{1}{2}\right)=\frac{1}{2}-\frac{\sqrt{3}}{6}\pi$

(5) $\displaystyle\int_{1-\sqrt{2}}^{1}\arctan x\,dx=\left[x\arctan x-\frac{1}{2}\log(1+x^2)\right]_{1-\sqrt{2}}^{1}=\left\{1\cdot\frac{\pi}{4}-(1-\sqrt{2})\cdot\left(-\frac{\pi}{8}\right)\right\}-\frac{1}{2}\left\{\log 2-\log(4-2\sqrt{2})\right\}$

$\displaystyle=\frac{3-\sqrt{2}}{8}\pi+\frac{1}{2}\log(2-\sqrt{2})$



⇩

$\arctan(1-\sqrt{2})=-\frac{\pi}{8}$

(6) $\displaystyle\int_{1}^{\sqrt{3}}\frac{1}{(1+x^2)(\arctan x)^3}dx=\int_{1}^{\sqrt{3}}(\arctan x)^{-3}\cdot\frac{1}{1+x^2}dx=\left[-\frac{1}{2}(\arctan x)^{-2}\right]_{1}^{\sqrt{3}}$

$\displaystyle=-\frac{1}{2}\left(\frac{9}{\pi^2}-\frac{16}{\pi^2}\right)=\frac{7}{2\pi^2}$

(7) $\displaystyle\int_{\frac{\sqrt{6}-\sqrt{2}}{4}}^{\frac{1}{\sqrt{2}}}\frac{\arccos x}{\sqrt{1-x^2}}dx=-\int_{\frac{\sqrt{6}-\sqrt{2}}{4}}^{\frac{1}{\sqrt{2}}}\arccos x\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right)dx=-\left[\frac{1}{2}(\arccos x)^2\right]_{\frac{\sqrt{6}-\sqrt{2}}{4}}^{\frac{1}{\sqrt{2}}}=-\frac{1}{2}\left(\frac{\pi^2}{16}-\frac{25}{144}\pi^2\right)=\frac{\pi^2}{18}$

※ $\displaystyle\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}-\frac{1}{2}\cdot\frac{1}{\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}$, $\displaystyle\frac{\pi}{6}+\frac{\pi}{4}=\frac{5}{12}\pi\in[0,\pi]$ より $\displaystyle\arccos\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{5}{12}\pi$

(8) $\displaystyle\int\log(1+x^2)dx=x\log(1+x^2)-\int\frac{2x^2}{1+x^2}dx=x\log(1+x^2)-\int\frac{2(1+x^2)-2}{1+x^2}dx$

$\displaystyle=x\log(1+x^2)-\int\left(2-\frac{2}{1+x^2}\right)dx=x\log(1+x^2)-2x+2\arctan x$

(9)
$$\begin{aligned}
\int_0^{\frac{1}{2}} \frac{1-x^3}{\sqrt{1-x^2}}\,dx &= \int_0^{\frac{1}{2}} \frac{1-x+(x-x^3)}{\sqrt{1-x^2}}\,dx = \int_0^{\frac{1}{2}} \left(\frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} + x\sqrt{1-x^2} \right) dx \\
&= \int_0^{\frac{1}{2}} \left\{ \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}}\cdot(-2x) - \frac{1}{2}(1-x^2)^{\frac{1}{2}}\cdot(-2x) \right\} dx \\
&= \left[\arcsin x + \frac{1}{2}\cdot 2(1-x^2)^{\frac{1}{2}} - \frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{\frac{3}{2}} \right]_0^{\frac{1}{2}} \\
&= \left(\frac{\pi}{6} - 0\right) + \left(\frac{\sqrt{3}}{2} - 1\right) - \frac{1}{3}\left(\frac{3\sqrt{3}}{8} - 1\right) = \frac{\pi}{6} + \frac{3\sqrt{3}}{8} - \frac{2}{3}
\end{aligned}$$

(10)
$$\begin{aligned}
\int \frac{2x\arctan x}{(1+x^2)^2}\,dx &= -\frac{\arctan x}{1+x^2} + \int \frac{1}{(1+x^2)^2}\,dx = -\frac{\arctan x}{1+x^2} + \int \frac{(1+x^2)-x^2}{(1+x^2)^2}\,dx \\
&= -\frac{\arctan x}{1+x^2} + \int \frac{1}{1+x^2}\,dx - \frac{1}{2}\int x\cdot\frac{2x}{(1+x^2)^2}\,dx \\
&= -\frac{\arctan x}{1+x^2} + \arctan x - \frac{1}{2}\left(-\frac{x}{1+x^2} + \int \frac{1}{1+x^2}\,dx \right) \\
&= -\frac{\arctan x}{1+x^2} + \frac{1}{2}\arctan x + \frac{x}{2(1+x^2)} \quad \left(= \frac{(x^2-1)\arctan x + x}{2(x^2+1)} \right)
\end{aligned}$$

$$\begin{array}{ccc}
\arctan x & & -\dfrac{1}{1+x^2} \\
\dfrac{1}{1+x^2} & & \dfrac{2x}{(1+x^2)^2}
\end{array}
\qquad
\begin{array}{ccc}
x & & -\dfrac{1}{1+x^2} \\
1 & & \dfrac{2x}{(1+x^2)^2}
\end{array}$$

$\boxed{2}$

(1)
$$\begin{aligned}
\sqrt{1+x-\frac{x^2}{4}} &= 1 + \frac{1}{2}\left(x-\frac{1}{4}x^2\right) - \frac{1}{8}\left(x-\frac{1}{4}x^2\right)^2 + \frac{1}{16}\left(x-\frac{1}{4}x^2\right)^3 - \frac{5}{128}\left(x-\frac{1}{4}x^2\right)^4 + \cdots \\
&= 1 + \frac{1}{2}\left(x-\frac{1}{4}x^2\right) - \frac{1}{8}\left(x^2-\frac{1}{2}x^3+\frac{1}{16}x^4\right) + \frac{1}{16}\left(x^3-\frac{3}{4}x^4+\cdots\right) - \frac{5}{128}\left(x^4+\cdots\right) + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{8}x^2 \\
&\qquad - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{1}{128}x^4 \\
&\qquad\qquad + \frac{1}{16}x^3 - \frac{3}{64}x^4 \\
&\qquad\qquad\qquad - \frac{5}{128}x^4 + \cdots \\
&= 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{1}{8}x^3 - \frac{3}{32}x^4 + \cdots
\end{aligned}$$

(2)
$$\begin{aligned}
e^{x\cos x} &= 1 + x\cos x + \frac{1}{2}(x\cos x)^2 + \frac{1}{6}(x\cos x)^3 + \frac{1}{24}(x\cos x)^4 + \frac{1}{120}(x\cos x)^5 + \cdots \\
&= 1 + \left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots\right) + \frac{1}{2}\left(x - \frac{1}{2}x^3 + \cdots\right)^2 + \frac{1}{6}\left(x - \frac{1}{2}x^3 + \cdots\right)^3 \\
&\qquad + \frac{1}{24}(x - \cdots)^4 + \frac{1}{120}(x - \cdots)^5 + \cdots \\
&= 1 + \left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 - \cdots\right) + \frac{1}{2}\left(x^2 - x^4 + \cdots\right) + \frac{1}{6}\left(x^3 - \frac{3}{2}x^5 + \cdots\right) \\
&\qquad + \frac{1}{24}\left(x^4 + \cdots\right) + \frac{1}{120}\left(x^5 + \cdots\right) + \cdots \\
&= 1 + x - \frac{1}{2}x^3 + \frac{1}{24}x^5 \\
&\qquad + \frac{1}{2}x^2 - \frac{1}{2}x^4 \\
&\qquad\qquad + \frac{1}{6}x^3 - \frac{1}{4}x^5 \\
&\qquad\qquad\qquad + \frac{1}{24}x^4 + \frac{1}{120}x^5 + \cdots \\
&= 1 + x + \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{11}{24}x^4 - \frac{1}{5}x^5 + \cdots
\end{aligned}$$