科目名	解析学A 解析学	対象	2OB	学部 研究科	理学部第一部	学科 専攻科		学籍 番号		評点
平成 28 年 7 月 28 日 (木) 3 回目 (〜 時限目)		担当	石川 学	学年		氏名				
試験 時間	60 分	注意 事項	① 筆記用具以外持込不可 ~~2. 下記のみ参照・持込可~~ ( )							

平成 28 年度前期定期試験

※解答用紙の裏面使用可

$\boxed{1}$ $(-\sqrt{3}+i)^{-1+\frac{i}{\pi}}$ を $a+bi$ $(a,b\in\mathbb{R})$ または $r(\cos\theta+i\sin\theta)$ $(r>0,\ \theta\in\mathbb{R})$ の形にせよ. (10 点)

$\boxed{2}$ 次の複素積分の値を求めよ. (60 点)

(1) $\displaystyle\int_C\left(\overline{z}+\frac{1}{2}\right)dz$ $\quad C: z=t^3-it^2\quad(-1\leqq t\leqq 2)$

(2) $\displaystyle\int_C\frac{\overline{z}}{\mathrm{Re}z}dz$ $\quad C: z=t-\frac{i}{t}\quad\left(\frac{1}{2}\leqq t\leqq 2\right)$

(3) $\displaystyle\int_C\frac{\mathrm{Re}z}{\mathrm{Im}z}dz$ $\quad C: z=\log t+it\quad(1\leqq t\leqq\sqrt{e})$

(4) $\displaystyle\int_C|z|^2dz$ $\quad C: z=t-i\sin t\quad\left(0\leqq t\leqq\frac{\pi}{2}\right)$

(5) $\displaystyle\int_C(\mathrm{Im}z)^2dz$ $\quad C: z=t^2+i\arctan t\quad(0\leqq t\leqq 1)$

3 $\boxed{\cancel{4}}$ 次の複素積分の値を求めよ. ただし, 積分経路は正の向きとする. (20 点)

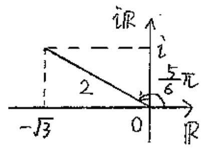
(1) $\displaystyle\int_{|z|=3}\frac{5z+7}{(z-2)(z+4)}dz$ $\quad$ (2) $\displaystyle\int_{|z|=2}\frac{5z-4}{z(z+1)(z-3)}dz$

(3) $\displaystyle\int_{|z|=4}\frac{-2z+1}{(z+2)(2z-5)(z-6)}dz$ $\quad$ (4) $\displaystyle\int_{|z|=1}\frac{1}{z^3(z+4)}dz$

東京理科大学　　年　月　日　**試験答案**　　評点

科目名		担当	先生
所属 学籍番号	学部　学科　番	氏名	

1



$$\log(-\sqrt{3}+i)=\log 2+i\left(\frac{5}{6}\pi+2m\pi\right)\quad (m\in\mathbb{Z})$$

だから

$$(-1+\tfrac{i}{\pi})\log(-\sqrt{3}+i)$$
$$=(-1+\tfrac{i}{\pi})\left\{\log 2+i\left(\tfrac{5}{6}\pi+2m\pi\right)\right\}$$
$$=-\log 2-\tfrac{5}{6}-2m+i\left(\tfrac{1}{\pi}\log 2-\tfrac{5}{6}\pi-2m\pi\right)$$

よって

$$(-\sqrt{3}+i)^{-1+\frac{i}{\pi}}$$
$$=e^{(-1+\frac{i}{\pi})\log(-\sqrt{3}+i)}$$
$$=e^{-\log 2-\frac{5}{6}-2m+i\left(\frac{1}{\pi}\log 2-\frac{5}{6}\pi-2m\pi\right)}$$
$$=e^{-\log 2-\frac{5}{6}-2m}\left\{\cos\left(\tfrac{1}{\pi}\log 2-\tfrac{5}{6}\pi-2m\pi\right)+i\sin\left(\tfrac{1}{\pi}\log 2-\tfrac{5}{6}\pi-2m\pi\right)\right\}$$
$$=\frac{1}{2}e^{-\frac{5}{6}-2m}\left\{\cos\left(\tfrac{1}{\pi}\log 2-\tfrac{5}{6}\pi\right)+i\sin\left(\tfrac{1}{\pi}\log 2-\tfrac{5}{6}\pi\right)\right\}\quad (m\in\mathbb{Z})$$

2

(1) $C: z=t^3-it^2\ (-1\leqq t\leqq 2)\ \rightarrow\ \frac{dz}{dt}=3t^2-2it$

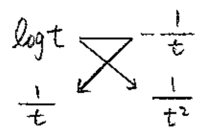
$$\int_C\left(\bar{z}+\frac{1}{2}\right)dz$$
$$=\int_{-1}^{2}\left(t^3+it^2+\tfrac{1}{2}\right)\cdot(3t^2-2it)\,dt$$
$$=\int_{-1}^{2}\left\{3t^5+2t^3+\tfrac{3}{2}t^2+i(t^4-t)\right\}dt$$
$$=\left[\tfrac{1}{2}t^6+\tfrac{1}{2}t^4+\tfrac{1}{2}t^3+i\left(\tfrac{1}{5}t^5-\tfrac{1}{2}t^2\right)\right]_{-1}^{2}$$
$$=(32+8+4)-\left(\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{2}\right)+i\left\{\left(\tfrac{32}{5}-2\right)-\left(-\tfrac{1}{5}-\tfrac{1}{2}\right)\right\}$$
$$=\frac{87}{2}+\frac{51}{10}i$$

(2) $C: z=t-\frac{i}{t}\ (\frac{1}{2}\leqq t\leqq 2)\ \rightarrow\ \frac{dz}{dt}=1+\frac{i}{t^2}$

$$\int_C\frac{\bar{z}}{\mathrm{Re}\,z}dz$$
$$=\int_{\frac{1}{2}}^{2}\frac{t+\frac{i}{t}}{t}\cdot\left(1+\frac{i}{t^2}\right)dt$$
$$=\int_{\frac{1}{2}}^{2}\left(1-\frac{1}{t^4}+\frac{2}{t^2}i\right)dt$$
$$=\left[t+\frac{1}{3t^3}-\frac{2}{t}i\right]_{\frac{1}{2}}^{2}$$
$$=\left(2+\tfrac{1}{24}\right)-\left(\tfrac{1}{2}+\tfrac{8}{3}\right)-i(1-4)$$
$$=-\frac{9}{8}+3i$$

(3) $C: z=\log t+it\ (1\leqq t\leqq\sqrt{e})\ \rightarrow\ \frac{dz}{dt}=\frac{1}{t}+i$

$$\int_C\frac{\mathrm{Re}\,z}{\mathrm{Im}\,z}dz$$
$$=\int_1^{\sqrt{e}}\frac{\log t}{t}\cdot\left(\frac{1}{t}+i\right)dt$$
$$=\int_1^{\sqrt{e}}\left(\frac{\log t}{t^2}+i\log t\cdot\frac{1}{t}\right)dt$$



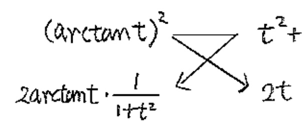
$$=\left[-\frac{\log t}{t}-\frac{1}{t}+\frac{i}{2}(\log t)^2\right]_1^{\sqrt{e}}$$
$$=\left(-\frac{\frac{1}{2}}{\sqrt{e}}-\frac{1}{\sqrt{e}}\right)-\left(-\frac{0}{1}-1\right)+\frac{i}{2}\left(\frac{1}{4}-0\right)$$
$$=-\frac{3}{2\sqrt{e}}+1+\frac{1}{8}i$$

(4) $C: z=t-i\sin t\ (0\leqq t\leqq\frac{\pi}{2})\ \rightarrow\ \frac{dz}{dt}=1-i\cos t$

$$\int_C|z|^2dz$$
$$=\int_0^{\frac{\pi}{2}}(t^2+\sin^2 t)\cdot(1-i\cos t)\,dt$$
$$=\int_0^{\frac{\pi}{2}}\left\{t^2+\sin^2 t-i(t^2\cos t+\sin^2 t\cdot\cos t)\right\}dt$$
$$=\left[\tfrac{1}{3}t^3\right]_0^{\frac{\pi}{2}}+\frac{1}{2}\cdot\frac{\pi}{2}-i\left[(t^2\sin t+2t\cos t-2\sin t)+\tfrac{1}{3}\sin^3 t\right]_0^{\frac{\pi}{2}}$$
$$=\frac{1}{3}\cdot\frac{\pi^3}{8}+\frac{\pi}{4}-i\left\{\left(\tfrac{\pi^2}{4}\cdot 1+\pi\cdot 0-2\cdot 1+\tfrac{1}{3}\cdot 1\right)-\left(0\cdot 0+0\cdot 1-2\cdot 0+\tfrac{1}{3}\cdot 0\right)\right\}$$
$$=\frac{\pi^3}{24}+\frac{\pi}{4}-i\left(\frac{\pi^2}{4}-\frac{5}{3}\right)$$

(5) $C: z=t^2+i\arctan t\ (0\leqq t\leqq 1)\ \rightarrow\ \frac{dz}{dt}=2t+\frac{i}{1+t^2}$

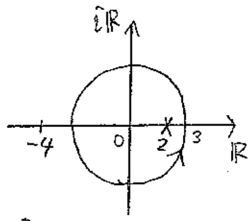
$$\int_C(\mathrm{Im}\,z)^2dz$$



$$=\int_0^1(\arctan t)^2\cdot\left(2t+\frac{i}{1+t^2}\right)dt$$
$$=\int_0^1\left\{2t(\arctan t)^2+i(\arctan t)^2\cdot\frac{1}{1+t^2}\right\}dt$$
$$=\left\{\left[(t^2+1)(\arctan t)^2\right]_0^1-2\int_0^1\arctan t\,dt\right\}+i\left[\tfrac{1}{3}(\arctan t)^3\right]_0^1$$
$$=\left(2\cdot\frac{\pi^2}{16}-1\cdot 0\right)-2\left[t\arctan t-\tfrac{1}{2}\log(1+t^2)\right]_0^1+\frac{i}{3}\left(\frac{\pi^3}{64}-0\right)$$
$$=\frac{\pi^2}{8}-2\left\{\left(1\cdot\tfrac{\pi}{4}-\tfrac{1}{2}\log 2\right)-\left(0\cdot 0-\tfrac{1}{2}\cdot 0\right)\right\}+\frac{\pi^3}{192}i$$
$$=\frac{\pi^2}{8}-\frac{\pi}{2}+\log 2+\frac{\pi^3}{192}i$$

$\boxed{3}$

(1) $\dfrac{5z+7}{(z-2)(z+4)} = \dfrac{A}{z-2} + \dfrac{B}{z+4} \cdots ①$



と分解できるから

$$\int_{|z|=3} \frac{5z+7}{(z-2)(z+4)}dz = \int_{|z|=3}\left(\frac{A}{z-2}+\frac{B}{z+4}\right)dz$$

$$= A\cdot 2\pi i$$

ここで，①より

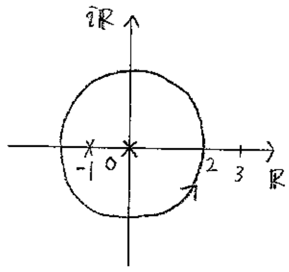
$$5z+7 = A(z+4)+B(z-2)$$

だから，$z=2$ を代入して

$$17 = 6A \qquad \therefore\ A = \frac{17}{6}$$

$$\therefore \int_{|z|=3}\frac{5z+7}{(z-2)(z+4)}dz = \frac{17}{6}\cdot 2\pi i = \frac{17}{3}\pi i$$

(2) $\dfrac{5z-4}{z(z+1)(z-3)} = \dfrac{A}{z}+\dfrac{B}{z+1}+\dfrac{C}{z-3} \cdots ①$



と分解できるから

$$\int_{|z|=2}\frac{5z-4}{z(z+1)(z-3)}dz$$

$$= \int_{|z|=2}\left(\frac{A}{z}+\frac{B}{z+1}+\frac{C}{z-3}\right)dz$$

$$= (A+B)\cdot 2\pi i$$

ここで，①より

$$5z-4 = A(z+1)(z-3)+Bz(z-3)+Cz(z+1)$$

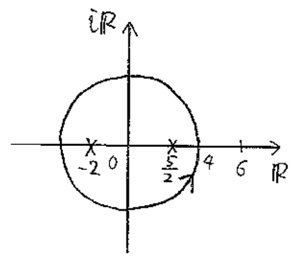
だから

$z=3$ を代入して　$11 = 12C$　$\therefore\ C = \dfrac{11}{12}$

z^2の係数を比較して　$0 = A+B+C$　$\therefore\ A+B = -\dfrac{11}{12}$

$$\therefore \int_{|z|=2}\frac{5z-4}{z(z+1)(z-3)}dz = -\frac{11}{12}\cdot 2\pi i = -\frac{11}{6}\pi i$$

(3) $\dfrac{-2z+1}{(z+2)(2z-5)(z-6)} = \dfrac{A}{z+2}+\dfrac{B}{2z-5}+\dfrac{C}{z-6} \cdots ①$



と分解できるから

$$\int_{|z|=4}\frac{-2z+1}{(z+2)(2z-5)(z-6)}dz$$

$$= \int_{|z|=4}\left(\frac{A}{z+2}+\frac{\frac{B}{2}}{z-\frac{5}{2}}+\frac{C}{z-6}\right)dz$$

$$= \left(A+\frac{B}{2}\right)\cdot 2\pi i$$

ここで，①より

$$-2z+1 = A(2z-5)(z-6)+B(z+2)(z-6)+C(z+2)(2z-5)$$

だから

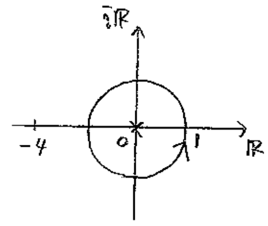
$z=6$ を代入して　$-11 = 56C$　$\therefore\ C = -\dfrac{11}{56}$

z^2の係数を比較して　$0 = 2A+B+2C$　$\therefore\ A+\dfrac{B}{2} = \dfrac{11}{56}$

$$\therefore \int_{|z|=4}\frac{-2z+1}{(z+2)(2z-5)(z-6)}dz = \frac{11}{56}\cdot 2\pi i = \frac{11}{28}\pi i$$

(4) $\dfrac{1}{z^3(z+4)} = \dfrac{A}{z}+\dfrac{B}{z^2}+\dfrac{C}{z^3}+\dfrac{D}{z+4} \cdots ①$

と分解できるから



$$\int_{|z|=1}\frac{1}{z^3(z+4)}dz$$

$$= \int_{|z|=1}\left(\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z^3}+\frac{D}{z+4}\right)dz$$

$$= A\cdot 2\pi i$$

ここで，①より

$$1 = (Az^2+Bz+C)(z+4)+Dz^3$$

だから

$z=-4$ を代入して　$1 = -64D$　$\therefore\ D = -\dfrac{1}{64}$

z^3の係数を比較して　$0 = A+D$　$\therefore\ A = \dfrac{1}{64}$

$$\therefore \int_{|z|=1}\frac{1}{z^3(z+4)}dz = \frac{1}{64}\cdot 2\pi i = \frac{\pi}{32}i$$