２００３年度基礎数学講義ノート（３－１組）

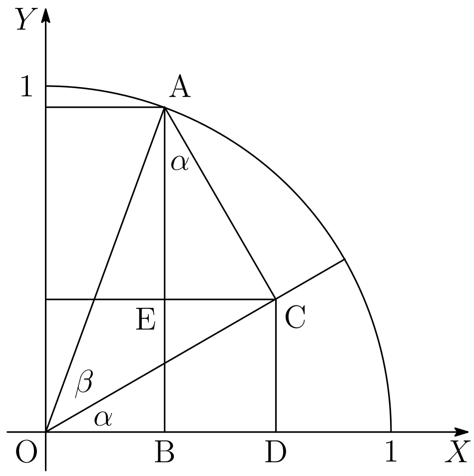
２００３年１０月４日分

６．加法定理

(1) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
(2) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
(3) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
(4) $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
(5) $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
(6) $\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

証明

$0 < \alpha, \beta, \alpha+\beta < \dfrac{\pi}{2}$ の場合のみ示す.



(1)
$$\begin{aligned}\sin(\alpha+\beta) &= \text{A の } Y \text{ 座標}\\ &= \mathrm{CD} + \mathrm{AE}\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta\end{aligned}$$

(3)
$$\begin{aligned}\cos(\alpha+\beta) &= \text{A の } X \text{ 座標}\\ &= \mathrm{OD} - \mathrm{CE}\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{aligned}$$

(5)
$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}\\ &= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}\\ &= \frac{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \quad \blacksquare\end{aligned}$$

$$\begin{cases}\mathrm{AC} = \sin\beta,\ \mathrm{OC} = \cos\beta\\ \mathrm{AE} = \mathrm{AC}\cos\alpha = \cos\alpha\sin\beta\\ \mathrm{CE} = \mathrm{AC}\sin\alpha = \sin\alpha\sin\beta\\ \mathrm{OD} = \mathrm{OC}\cos\alpha = \cos\alpha\cos\beta\\ \mathrm{CD} = \mathrm{OC}\sin\alpha = \sin\alpha\cos\beta\end{cases}$$

７．加法定理の応用

(1) 2 倍角の公式

$$\begin{cases}\sin 2x = 2\sin x\cos x\\ \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x\\ \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}\end{cases}$$

証明

$\sin 2x = \sin(x+x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$
$\cos 2x = \cos(x+x) = \cos x\cos x - \sin x\sin x = \cos^2 x - \sin^2 x$
$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1-\cos^2 x) = 2\cos^2 x - 1$
$\cos 2x = \cos^2 x - \sin^2 x = (1-\sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$
$\tan 2x = \tan(x+x) = \dfrac{\tan x+\tan x}{1-\tan x\tan x} = \dfrac{2\tan x}{1-\tan^2 x}$ $\blacksquare$

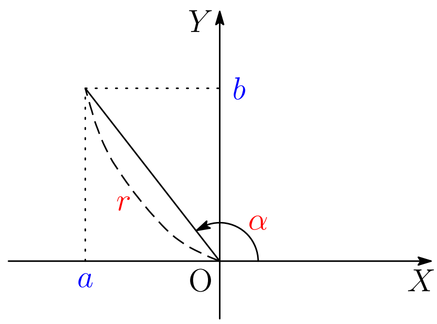
(4) 三角関数の合成

$$a\sin x + b\cos x = r\sin(x+\alpha)$$

ここで，r と α は右図のようなもので，

$$\begin{cases} r = \sqrt{a^2+b^2} \\ \cos\alpha = \dfrac{a}{r}, \ \sin\alpha = \dfrac{b}{r} \end{cases}$$

を満たす．



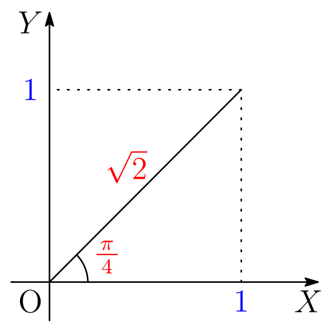
証明

$r\sin(x+\alpha) = r(\sin x\cos\alpha + \cos x\sin\alpha) = r\cos\alpha\cdot\sin x + r\sin\alpha\cdot\cos x = a\sin x + b\cos x$ ∎

問題４．５

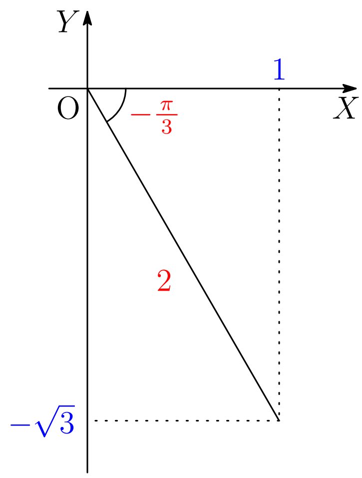
(1) 右図より

$$\begin{aligned} \sin x + \cos x &= 1\cdot\sin x + 1\cdot\cos x \\ &= \sqrt{2}\sin\left(x+\frac{\pi}{4}\right) \end{aligned}$$



(4) 右図より

$$\begin{aligned} \sin x - \sqrt{3}\cos x &= 1\cdot\sin x - \sqrt{3}\cdot\cos x \\ &= 2\sin\left(x-\frac{\pi}{3}\right) \end{aligned}$$



問題４．８

(1) $y = \sin x - \cos x \quad (0 \leqq x \leqq \pi)$

上と同様に図示すれば

$$y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$$

が分かる．右図より

$$-\frac{1}{\sqrt{2}} \leqq \sin\left(x - \frac{\pi}{4}\right) \leqq 1$$

なので

$$-1 \leqq y \leqq \sqrt{2}$$

よって

$$\begin{cases} x - \dfrac{\pi}{4} = \dfrac{\pi}{2} \text{ 即ち } x = \dfrac{3}{4}\pi \text{ のとき最大値 } \sqrt{2} \\ x - \dfrac{\pi}{4} = -\dfrac{\pi}{4} \text{ 即ち } x = 0 \text{ のとき最小値 } -1 \end{cases}$$