２００３年度基礎数学講義ノート（２－１組）

２００３年１２月１３日分

行列の対角化

A：n 次正方行列
$\lambda_1, \ldots, \lambda_n$：$A$ の固有値
$\boldsymbol{p}_1, \ldots, \boldsymbol{p}_n$：それぞれ $\lambda_1, \ldots, \lambda_n$ に対する A の固有ベクトル
このとき

$$A\boldsymbol{p}_1 = \lambda_1\boldsymbol{p}_1, \quad \ldots \quad , A\boldsymbol{p}_n = \lambda_n\boldsymbol{p}_n$$

なので $P = (\boldsymbol{p}_1 \ \cdots \ \boldsymbol{p}_n)$ とおけば P は n 次正方行列で

$$AP = (A\boldsymbol{p}_1 \ \cdots \ A\boldsymbol{p}_n) = (\lambda_1\boldsymbol{p}_1 \ \cdots \ \lambda_n\boldsymbol{p}_n) = P\begin{pmatrix} \lambda_1 & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & \lambda_n \end{pmatrix}$$

よって，P が正則ならば P^{-1} を左からかけて

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & \lambda_n \end{pmatrix}$$

例（前回の例に関連）

(1) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \qquad \lambda = -1 \cdots \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ \lambda = 4 \cdots \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ とおくと $P^{-1}AP = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$

(2) $A = \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \qquad \lambda = 1 \cdots \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \lambda = 2 \cdots \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \ \lambda = 3 \cdots \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ とおくと $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(3) $A = \begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \qquad \lambda = 1 \cdots \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $\lambda = -1$（2 重解）$\cdots \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$P = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$ とおくと $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

確認

(1) $P^{-1} = \dfrac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}$（p.8 参照）なので

$$P^{-1}AP = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 8 \\ 1 & 12 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$(2)\ (P \mid E) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-R_1]{R_2-R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[-R_3]{-R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{R_1-R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right) \qquad \therefore\ P^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$(3)\ (P \mid E) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+R_1]{R_2+R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right)$$

$$\xrightarrow[R_2-R_3]{R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -1 \end{array}\right) \qquad \therefore\ P^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} -3 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$