1　次の極限値を求めよ．ただし，(7) において $[x]$ はガウス記号（x をこえない最大の整数）である．

(1) $\displaystyle\lim_{n\to\infty}\frac{2n^2-3n+5}{9n^2+2n+1}$

(2) $\displaystyle\lim_{n\to\infty}\frac{(7n^3+1)(n^2+n-4)}{(n+2)^3(5n+3)^2}$

(3) $\displaystyle\lim_{n\to\infty}\frac{({}_{2n}\mathrm{C}_2)^2}{{}_{4n}\mathrm{C}_4}$

(4) $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n^2+3n}-\sqrt{n^2-1}}$

(5) $\displaystyle\lim_{n\to\infty}\frac{\sqrt{3n+5}-\sqrt{3n-2}}{\sqrt{4n+2}-\sqrt{4n-1}}$

(6) $\displaystyle\lim_{n\to\infty}n\left(\sqrt{n^2-4n+13}-n+2\right)$

(7)* $\displaystyle\lim_{n\to\infty}\left(\sqrt{9n^2+10n+11}-\left[\sqrt{9n^2+10n+11}\right]\right)$

2　次の和を求めよ．ただし，(5) では $\lim_{n\to\infty} np^n = 0\ (-1 < p < 1)$ を用いてよい．

(1) $\displaystyle\sum_{n=1}^{\infty}\left\{\left(-\frac{2}{3}\right)^{n-1}+\left(-\frac{5}{8}\right)^{n}\right\}$

(2) $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{3}{4^{n-1}}-\frac{(-2)^n}{3^{n-1}}\right\}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{4\cdot 5^n-5\cdot 8^n}{12^n}$

(4) $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{3}{2^n}-\frac{5}{(-3)^n}+\frac{7}{(-5)^n}-\frac{2}{7^n}\right\}$

(5) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{6^n}$

3　次の和を求めよ．

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)}$

(2) $\displaystyle\sum_{n=1}^{\infty}\frac{10}{(5n-2)(5n+3)}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{8}{(2n-1)(2n+3)(2n+7)}$

1 次の極限値を求めよ．ただし，(7) において $[x]$ はガウス記号（x をこえない最大の整数）である．

(1) $\displaystyle\lim_{n\to\infty}\frac{2n^2-3n+5}{9n^2+2n+1}$　　(2) $\displaystyle\lim_{n\to\infty}\frac{(7n^3+1)(n^2+n-4)}{(n+2)^3(5n+3)^2}$

(3) $\displaystyle\lim_{n\to\infty}\frac{({}_{2n}\mathrm{C}_2)^2}{{}_{4n}\mathrm{C}_4}$　　(4) $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n^2+3n}-\sqrt{n^2-1}}$

(5) $\displaystyle\lim_{n\to\infty}\frac{\sqrt{3n+5}-\sqrt{3n-2}}{\sqrt{4n+2}-\sqrt{4n-1}}$　　(6) $\displaystyle\lim_{n\to\infty}n\left(\sqrt{n^2-4n+13}-n+2\right)$

(7)* $\displaystyle\lim_{n\to\infty}\left(\sqrt{9n^2+10n+11}-\left[\sqrt{9n^2+10n+11}\right]\right)$

解答

(1) $$\lim_{n\to\infty}\frac{2n^2-3n+5}{9n^2+2n+1}=\lim_{n\to\infty}\frac{2-\frac{3}{n}+\frac{5}{n^2}}{9+\frac{2}{n}+\frac{1}{n^2}}=\frac{2}{9}$$

(2) $$\lim_{n\to\infty}\frac{(7n^3+1)(n^2+n-4)}{(n+2)^3(5n+3)^2}=\lim_{n\to\infty}\frac{\left(7+\frac{1}{n^3}\right)\left(1+\frac{1}{n}-\frac{4}{n^2}\right)}{\left(1+\frac{2}{n}\right)^3\left(5+\frac{3}{n}\right)^2}=\frac{7\cdot1}{1^3\cdot5^2}=\frac{7}{25}$$

(3) $$\lim_{n\to\infty}\frac{({}_{2n}\mathrm{C}_2)^2}{{}_{4n}\mathrm{C}_4}=\lim_{n\to\infty}\frac{\left\{\frac{2n(2n-1)}{2\cdot1}\right\}^2}{\frac{4n(4n-1)(4n-2)(4n-3)}{4\cdot3\cdot2\cdot1}}=\lim_{n\to\infty}\frac{3n(2n-1)}{(4n-1)(4n-3)}$$

$$=\lim_{n\to\infty}\frac{3\left(2-\frac{1}{n}\right)}{\left(4-\frac{1}{n}\right)\left(4-\frac{3}{n}\right)}=\frac{3\cdot2}{4\cdot4}=\frac{3}{8}$$

(4) $$\lim_{n\to\infty}\frac{1}{\sqrt{n^2+3n}-\sqrt{n^2-1}}=\lim_{n\to\infty}\frac{\sqrt{n^2+3n}+\sqrt{n^2-1}}{(n^2+3n)-(n^2-1)}=\lim_{n\to\infty}\frac{\sqrt{n^2+3n}+\sqrt{n^2-1}}{3n+1}$$

$$=\lim_{n\to\infty}\frac{\sqrt{1+\frac{3}{n}}+\sqrt{1-\frac{1}{n^2}}}{3+\frac{1}{n}}=\frac{1+1}{3}=\frac{2}{3}$$

(5) $$\lim_{n\to\infty}\frac{\sqrt{3n+5}-\sqrt{3n-2}}{\sqrt{4n+2}-\sqrt{4n-1}}=\lim_{n\to\infty}\frac{\{(3n+5)-(3n-2)\}(\sqrt{4n+2}+\sqrt{4n-1})}{\{(4n+2)-(4n-1)\}(\sqrt{3n+5}+\sqrt{3n-2})}$$

$$=\lim_{n\to\infty}\frac{7(\sqrt{4n+2}+\sqrt{4n-1})}{3(\sqrt{3n+5}+\sqrt{3n-2})}=\lim_{n\to\infty}\frac{7\left(\sqrt{4+\frac{2}{n}}+\sqrt{4-\frac{1}{n}}\right)}{3\left(\sqrt{3+\frac{5}{n}}+\sqrt{3-\frac{2}{n}}\right)}$$

$$=\frac{7(2+2)}{3(\sqrt{3}+\sqrt{3})}=\frac{14}{3\sqrt{3}}$$

(6) $\displaystyle\lim_{n\to\infty} n\left(\sqrt{n^2-4n+13}-n+2\right)=\lim_{n\to\infty}\frac{n\{(n^2-4n+13)-(n-2)^2\}}{\sqrt{n^2-4n+13}+(n-2)}$

$\displaystyle=\lim_{n\to\infty}\frac{9n}{\sqrt{n^2-4n+13}+n-2}=\lim_{n\to\infty}\frac{9}{\sqrt{1-\frac{4}{n}+\frac{13}{n^2}}+1-\frac{2}{n}}=\frac{9}{1+1}=\frac{9}{2}$

(7)* $(9n^2+10n+11)-(3n+1)^2=4n+10>0,\ (3n+2)^2-(9n^2+10n+11)=2n-7>0$
（ただし，第 2 式は $n\geqq 4$ のとき）より

$3n+1<\sqrt{9n^2+10n+11}<3n+2 \qquad \therefore\ \left[\sqrt{9n^2+10n+11}\right]=3n+1 \quad (n\geqq 4)$

よって

$\displaystyle\lim_{n\to\infty}\left(\sqrt{9n^2+10n+11}-\left[\sqrt{9n^2+10n+11}\right]\right)=\lim_{n\to\infty}\left\{\sqrt{9n^2+10n+11}-(3n+1)\right\}$

$\displaystyle=\lim_{n\to\infty}\frac{(9n^2+10n+11)-(3n+1)^2}{\sqrt{9n^2+10n+11}+(3n+1)}=\lim_{n\to\infty}\frac{4n+10}{\sqrt{9n^2+10n+11}+3n+1}$

$\displaystyle=\lim_{n\to\infty}\frac{4+\frac{10}{n}}{\sqrt{9+\frac{10}{n}+\frac{11}{n^2}}+3+\frac{1}{n}}=\frac{4}{3+3}=\frac{2}{3}$

※ $9n^2+10n+11=\left(3n+\frac{5}{3}\right)^2+\frac{74}{9}$ より，n が十分大きいとき $\sqrt{9n^2+10n+11}\doteqdot 3n+\frac{5}{3}$ である．これから，$3n+1<\sqrt{9n^2+10n+11}<3n+2$ が成り立つことが予想できる．ただし，これは予想であって，正しいことを確認しなければいけない．また，$n\to\infty$ のときを考えるから，$n\geqq 1$ で成り立つ必要はなく，十分大きい n について成り立てばよい．

2 次の和を求めよ．ただし，(5) では $\lim_{n\to\infty} np^n = 0\ (-1 < p < 1)$ を用いてよい．

(1) $\displaystyle\sum_{n=1}^{\infty}\left\{\left(-\frac{2}{3}\right)^{n-1}+\left(-\frac{5}{8}\right)^{n}\right\}$

(2) $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{3}{4^{n-1}}-\frac{(-2)^n}{3^{n-1}}\right\}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{4\cdot 5^n-5\cdot 8^n}{12^n}$

(4) $\displaystyle\sum_{n=1}^{\infty}\left\{\frac{3}{2^n}-\frac{5}{(-3)^n}+\frac{7}{(-5)^n}-\frac{2}{7^n}\right\}$

(5) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{6^n}$

解答

(1) $$\sum_{n=1}^{\infty}\left\{\left(-\frac{2}{3}\right)^{n-1}+\left(-\frac{5}{8}\right)^{n}\right\}=\frac{1}{1-\left(-\frac{2}{3}\right)}+\frac{-\frac{5}{8}}{1-\left(-\frac{5}{8}\right)}=\frac{3}{5}-\frac{5}{13}=\frac{14}{65}$$

(2) $$\sum_{n=1}^{\infty}\left\{\frac{3}{4^{n-1}}-\frac{(-2)^n}{3^{n-1}}\right\}=\sum_{n=1}^{\infty}\left\{3\cdot\left(\frac{1}{4}\right)^{n-1}+2\cdot\left(-\frac{2}{3}\right)^{n-1}\right\}=\frac{3}{1-\frac{1}{4}}+\frac{2}{1-\left(-\frac{2}{3}\right)}$$

$$=4+\frac{6}{5}=\frac{26}{5}$$

(3) $$\sum_{n=1}^{\infty}\frac{4\cdot 5^n-5\cdot 8^n}{12^n}=\sum_{n=1}^{\infty}\left\{4\cdot\left(\frac{5}{12}\right)^{n}-5\cdot\left(\frac{2}{3}\right)^{n}\right\}=\frac{\frac{20}{12}}{1-\frac{5}{12}}-\frac{\frac{10}{3}}{1-\frac{2}{3}}$$

$$=\frac{20}{7}-10=-\frac{50}{7}$$

(4) $$\sum_{n=1}^{\infty}\left\{\frac{3}{2^n}-\frac{5}{(-3)^n}+\frac{7}{(-5)^n}-\frac{2}{7^n}\right\}$$

$$=\sum_{n=1}^{\infty}\left\{3\cdot\left(\frac{1}{2}\right)^{n}-5\cdot\left(-\frac{1}{3}\right)^{n}+7\cdot\left(-\frac{1}{5}\right)^{n}-2\cdot\left(\frac{1}{7}\right)^{n}\right\}$$

$$=\frac{\frac{3}{2}}{1-\frac{1}{2}}-\frac{-\frac{5}{3}}{1-\left(-\frac{1}{3}\right)}+\frac{-\frac{7}{5}}{1-\left(-\frac{1}{5}\right)}-\frac{\frac{2}{7}}{1-\frac{1}{7}}=3+\frac{5}{4}-\frac{7}{6}-\frac{1}{3}=\frac{11}{4}$$

(5) 部分和を $S_n = \sum_{k=1}^{n} \frac{k}{6^k}$ とおくと

$$\begin{array}{rrcccccccccc}
 & S_n & = & \frac{1}{6} & + & \frac{2}{6^2} & + & \cdots & + & \frac{n}{6^n} & & \\
-) & \frac{1}{6}S_n & = & & & \frac{1}{6^2} & + & \cdots & + & \frac{n-1}{6^n} & + & \frac{n}{6^{n+1}} \\
\hline
 & \frac{5}{6}S_n & = & \frac{1}{6} & + & \frac{1}{6^2} & + & \cdots & + & \frac{1}{6^n} & - & \frac{n}{6^{n+1}}
\end{array}$$

$$\begin{aligned}
&= \sum_{k=1}^{n} \frac{1}{6^k} - \frac{n}{6^{n+1}} \\
&= \frac{\frac{1}{6}\left(1 - \frac{1}{6^n}\right)}{1 - \frac{1}{6}} - \frac{n}{6^{n+1}} \\
&= \frac{1}{5}\left(1 - \frac{1}{6^n}\right) - \frac{n}{6^{n+1}}
\end{aligned}$$

であるから

$$S_n = \frac{6}{25}\left(1 - \frac{1}{6^n}\right) - \frac{n}{5 \cdot 6^n}$$

よって

$$\sum_{n=1}^{\infty} \frac{n}{6^n} = \lim_{n \to \infty} \left\{ \frac{6}{25}\left(1 - \frac{1}{6^n}\right) - \frac{n}{5 \cdot 6^n} \right\} = \frac{6}{25}$$

3　次の和を求めよ．

(1) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)}$　　(2) $\displaystyle\sum_{n=1}^{\infty}\frac{10}{(5n-2)(5n+3)}$

(3) $\displaystyle\sum_{n=1}^{\infty}\frac{8}{(2n-1)(2n+3)(2n+7)}$

解答

(1) 部分和は

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{(k+3)(k+4)} &= \sum_{k=1}^{n}\left(\frac{1}{k+3}-\frac{1}{k+4}\right)\\ &= \sum_{k=1}^{n}\frac{1}{k+3}-\sum_{k=1}^{n}\frac{1}{k+4}\\ &= \left(\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{n+3}\right)\\ &\qquad -\left(\frac{1}{5}+\cdots+\frac{1}{n+3}+\frac{1}{n+4}\right)\\ &= \frac{1}{4}-\frac{1}{n+4}\end{aligned}$$

であるから

$$\sum_{n=1}^{\infty}\frac{1}{(n+3)(n+4)}=\lim_{n\to\infty}\left(\frac{1}{4}-\frac{1}{n+4}\right)=\frac{1}{4}$$

※ $\dfrac{1}{(k+3)(k+4)}=\dfrac{(k+4)-(k+3)}{(k+3)(k+4)}=\dfrac{k+4}{(k+3)(k+4)}-\dfrac{k+3}{(k+3)(k+4)}=\dfrac{1}{k+3}-\dfrac{1}{k+4}$

(2) 部分和は

$$\begin{aligned}\sum_{k=1}^{n}\frac{10}{(5k-2)(5k+3)} &= \sum_{k=1}^{n}\left(\frac{2}{5k-2}-\frac{2}{5k+3}\right)\\ &= \sum_{k=1}^{n}\frac{2}{5k-2}-\sum_{k=1}^{n}\frac{2}{5k+3}\\ &= \left(\frac{2}{3}+\frac{2}{8}+\cdots+\frac{2}{5n-2}\right)\\ &\qquad -\left(\frac{2}{8}+\cdots+\frac{2}{5n-2}+\frac{2}{5n+3}\right)\\ &= \frac{2}{3}-\frac{2}{5n+3}\end{aligned}$$

であるから

$$\sum_{n=1}^{\infty}\frac{10}{(5n-2)(5n+3)}=\lim_{n\to\infty}\left(\frac{2}{3}-\frac{2}{5n+3}\right)=\frac{2}{3}$$

$$※ \frac{10}{(5k-2)(5k+3)} = \frac{(5k+3)-(5k-2)}{(5k-2)(5k+3)} \cdot 2$$
$$= 2\left\{\frac{5k+3}{(5k-2)(5k+3)} - \frac{5k-2}{(5k-2)(5k+3)}\right\}$$
$$= \frac{2}{5k-2} - \frac{2}{5k+3}$$

(3) 部分和は

$$\sum_{k=1}^{n} \frac{8}{(2k-1)(2k+3)(2k+7)}$$
$$= \sum_{k=1}^{n} \left\{\frac{1}{(2k-1)(2k+3)} - \frac{1}{(2k+3)(2k+7)}\right\}$$
$$= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+3)} - \sum_{k=1}^{n} \frac{1}{(2k+3)(2k+7)}$$
$$= \left\{\frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 9} + \cdots + \frac{1}{(2n-1)(2n+3)}\right\}$$
$$- \left\{\frac{1}{5\cdot 9} + \cdots + \frac{1}{(2n-1)(2n+3)} + \frac{1}{(2n+1)(2n+5)} + \frac{1}{(2n+3)(2n+7)}\right\}$$
$$= \frac{1}{1\cdot 5} + \frac{1}{3\cdot 7} - \frac{1}{(2n+1)(2n+5)} - \frac{1}{(2n+3)(2n+7)}$$
$$= \frac{26}{105} - \frac{1}{(2n+1)(2n+5)} - \frac{1}{(2n+3)(2n+7)}$$

であるから

$$\sum_{n=1}^{\infty} \frac{8}{(2n-1)(2n+3)(2n+7)} = \lim_{n\to\infty}\left\{\frac{26}{105} - \frac{1}{(2n+1)(2n+5)} - \frac{1}{(2n+3)(2n+7)}\right\} = \frac{26}{105}$$

$$※ \frac{8}{(2k-1)(2k+3)(2k+7)} = \frac{(2k+7)-(2k-1)}{(2k-1)(2k+3)(2k+7)}$$
$$= \frac{2k+7}{(2k-1)(2k+3)(2k+7)} - \frac{2k-1}{(2k-1)(2k+3)(2k+7)}$$
$$= \frac{1}{(2k-1)(2k+3)} - \frac{1}{(2k+3)(2k+7)}$$