積分計算（３）

三角関数の積分

C は積分定数とする．

(1) $\displaystyle\int \sin x dx = -\cos x + C$　　(2) $\displaystyle\int \cos x dx = \sin x + C$

(3) $\displaystyle\int \frac{1}{\cos^2 x} dx = \tan x + C$　　(4) $\displaystyle\int \frac{1}{\sin^2 x} dx = -\frac{1}{\tan x} + C$

※1 次関数との合成
定数 $A\ (\neq 0)$，B に対して

(1) $\displaystyle\int \sin(Ax+B)dx = -\frac{1}{A}\cos(Ax+B) + C$

(2) $\displaystyle\int \cos(Ax+B)dx = \frac{1}{A}\sin(Ax+B) + C$

(3) $\displaystyle\int \frac{1}{\cos^2(Ax+B)} dx = \frac{1}{A}\tan(Ax+B) + C$

(4) $\displaystyle\int \frac{1}{\sin^2(Ax+B)} dx = -\frac{1}{A\tan(Ax+B)} + C$

※三角関数の積分で頻繁に用いる公式

(1) $\sin^2 x = \dfrac{1-\cos 2x}{2}$

(2) $\cos^2 x = \dfrac{1+\cos 2x}{2}$

(3) $\sin\alpha\cos\beta = \dfrac{1}{2}\{\sin(\alpha+\beta)+\sin(\alpha-\beta)\}$

(4) $\cos\alpha\sin\beta = \dfrac{1}{2}\{\sin(\alpha+\beta)-\sin(\alpha-\beta)\}$

(5) $\cos\alpha\cos\beta = \dfrac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$

(6) $\sin\alpha\sin\beta = -\dfrac{1}{2}\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}$

【問題 3】
次の原始関数（不定積分）を求めよ．

(1) $\displaystyle\int \frac{3+2\cos^3 x}{\cos^2 x} dx$　　(2) $\displaystyle\int \frac{1-\cos x\sin^2 x}{\sin^2 x} dx$

(3) $\displaystyle\int \frac{1}{\tan^2 x} dx$　　(4) $\displaystyle\int \left(\tan x + \frac{3}{\tan x}\right)^2 dx$

(5) $\displaystyle\int \left(2\tan x - \frac{3}{\tan x}\right)^2 dx$　　(6) $\displaystyle\int \left(\frac{2\tan x}{3} + \frac{5}{2\tan x}\right)^2 dx$

(7) $\displaystyle\int \cos(3x-1)dx$

(8) $\displaystyle\int \frac{1}{\cos^2(2x-1)}dx$

(9) $\displaystyle\int (3\sin 2x - 2\cos 4x)dx$

(10) $\displaystyle\int (\sin x + \cos x)^2 dx$

(11) $\displaystyle\int \left(\cos\frac{3x}{2} + \sin\frac{x}{4} + \frac{3}{\sin^2 5x}\right)dx$

(12) $\displaystyle\int \left\{2\cos(3x+5) - \frac{1}{2\cos^2(3x+5)}\right\}dx$

(13) $\displaystyle\int \cos^2 x dx$

(14) $\displaystyle\int \sin^3 x dx$

(15) $\displaystyle\int \cos^4 x dx$

(16) $\displaystyle\int \cos 4x \cos x dx$

(17) $\displaystyle\int \cos 3x \cos 2x dx$

(18) $\displaystyle\int \cos 3x \cos 5x dx$

(19) $\displaystyle\int \sin 3x \sin x dx$

(20) $\displaystyle\int \sin 4x \sin 6x dx$

(21) $\displaystyle\int \sin 3x \cos 2x dx$

(22) $\displaystyle\int \sin 2x \cos 4x dx$

(23) $\displaystyle\int (2\sin 5x - 3\cos 3x)^2 dx$

(24) $\displaystyle\int (2\cos 2x - \cos 5x)^2 dx$

(25) $\displaystyle\int (2\sin 3x - \cos 4x)^2 dx$

(26) $\displaystyle\int (2\sin 6x - 3\cos 2x)^2 dx$

(27) $\displaystyle\int (2\cos 7x - 5\sin 3x)^2 dx$

(28) $\displaystyle\int (2\cos 7x - 5\cos 4x)^2 dx$

(29) $\displaystyle\int (2\sin 5x - 3\cos 2x + 1)^2 dx$

解答

(1) $\displaystyle\int \frac{3+2\cos^3 x}{\cos^2 x}dx = \int\left(\frac{3}{\cos^2 x} + 2\cos x\right)dx = 3\tan x + 2\sin x + C$

(2) $\displaystyle\int \frac{1-\cos x\sin^2 x}{\sin^2 x}dx = \int\left(\frac{1}{\sin^2 x} - \cos x\right)dx = -\frac{1}{\tan x} - \sin x + C$

(3) $\displaystyle\int \frac{1}{\tan^2 x}dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx = -\frac{1}{\tan x} - x + C$

(4) $\displaystyle\int \left(\tan x + \frac{3}{\tan x}\right)^2 dx = \int\left(\tan^2 x + 6 + \frac{9}{\tan^2 x}\right)dx$

$\displaystyle= \int\left\{\left(\frac{1}{\cos^2 x} - 1\right) + 6 + 9\left(\frac{1}{\sin^2 x} - 1\right)\right\}dx = \int\left(\frac{1}{\cos^2 x} + \frac{9}{\sin^2 x} - 4\right)dx$

$\displaystyle= \tan x - \frac{9}{\tan x} - 4x + C$

(5) $$\int\left(2\tan x-\frac{3}{\tan x}\right)^2dx=\int\left(4\tan^2 x-12+\frac{9}{\tan^2 x}\right)dx$$
$$=\int\left\{4\left(\frac{1}{\cos^2 x}-1\right)-12+9\left(\frac{1}{\sin^2 x}-1\right)\right\}dx$$
$$=\int\left(\frac{4}{\cos^2 x}+\frac{9}{\sin^2 x}-25\right)dx=4\tan x-\frac{9}{\tan x}-25x+C$$

(6) $$\int\left(\frac{2\tan x}{3}+\frac{5}{2\tan x}\right)^2dx=\int\left(\frac{4}{9}\tan^2 x+\frac{10}{3}+\frac{25}{4\tan^2 x}\right)dx$$
$$=\int\left\{\frac{4}{9}\left(\frac{1}{\cos^2 x}-1\right)+\frac{10}{3}+\frac{25}{4}\left(\frac{1}{\sin^2 x}-1\right)\right\}dx$$
$$=\int\left(\frac{4}{9\cos^2 x}+\frac{25}{4\sin^2 x}-\frac{121}{36}\right)dx=\frac{4}{9}\tan x-\frac{25}{4\tan x}-\frac{121}{36}x+C$$

(7) $$\int\cos(3x-1)dx=\frac{1}{3}\sin(3x-1)+C$$

(8) $$\int\frac{1}{\cos^2(2x-1)}dx=\frac{1}{2}\tan(2x-1)+C$$

(9) $$\int(3\sin 2x-2\cos 4x)dx=-\frac{3}{2}\cos 2x-\frac{1}{2}\sin 4x+C$$

(10) $$\int(\sin x+\cos x)^2dx=\int(\sin^2 x+2\sin x\cos x+\cos^2 x)dx=\int(1+\sin 2x)dx$$
$$=x-\frac{1}{2}\cos 2x+C$$

(11) $$\int\left(\cos\frac{3x}{2}+\sin\frac{x}{4}+\frac{3}{\sin^2 5x}\right)dx=\frac{2}{3}\sin\frac{3x}{2}-4\cos\frac{x}{4}-\frac{3}{5\tan 5x}+C$$

(12) $$\int\left\{2\cos(3x+5)-\frac{1}{2\cos^2(3x+5)}\right\}dx=\frac{2}{3}\sin(3x+5)-\frac{1}{6}\tan(3x+5)+C$$

(13) $$\int\cos^2 x dx=\int\frac{1+\cos 2x}{2}dx=\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+C=\frac{1}{2}x+\frac{1}{4}\sin 2x+C$$

(14) $$\int\sin^3 x dx=\int\frac{3\sin x-\sin 3x}{4}dx=\frac{1}{4}\left(-3\cos x+\frac{1}{3}\cos 3x\right)+C$$
$$=-\frac{3}{4}\cos x+\frac{1}{12}\cos 3x+C$$

(15) $$\int\cos^4 x dx=\int\left(\frac{1+\cos 2x}{2}\right)^2dx=\int\frac{1}{4}(1+2\cos 2x+\cos^2 2x)dx$$
$$=\int\frac{1}{4}\left(1+2\cos 2x+\frac{1+\cos 4x}{2}\right)dx=\int\left(\frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\right)dx$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

(16) $$\int \cos 4x \cos x dx = \int \frac{1}{2}(\cos 5x + \cos 3x)dx = \frac{1}{2}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x\right) + C$$

$$= \frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + C$$

(17) $$\int \cos 3x \cos 2x dx = \int \frac{1}{2}(\cos 5x + \cos x)dx = \frac{1}{2}\left(\frac{1}{5}\sin 5x + \sin x\right) + C$$

$$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C$$

(18) $$\int \cos 3x \cos 5x dx = \int \cos 5x \cos 3x dx = \int \frac{1}{2}(\cos 8x + \cos 2x)dx$$

$$= \frac{1}{2}\left(\frac{1}{8}\sin 8x + \frac{1}{2}\sin 2x\right) + C = \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + C$$

(19) $$\int \sin 3x \sin x dx = \int \left\{-\frac{1}{2}(\cos 4x - \cos 2x)\right\} dx = -\frac{1}{2}\left(\frac{1}{4}\sin 4x - \frac{1}{2}\sin 2x\right) + C$$

$$= -\frac{1}{8}\sin 4x + \frac{1}{4}\sin 2x + C$$

(20) $$\int \sin 4x \sin 6x dx = \int \sin 6x \sin 4x dx = \int \left\{-\frac{1}{2}(\cos 10x - \cos 2x)\right\} dx$$

$$= -\frac{1}{2}\left(\frac{1}{10}\sin 10x - \frac{1}{2}\sin 2x\right) + C = -\frac{1}{20}\sin 10x + \frac{1}{4}\sin 2x + C$$

(21) $$\int \sin 3x \cos 2x dx = \int \frac{1}{2}(\sin 5x + \sin x)dx = \frac{1}{2}\left(-\frac{1}{5}\cos 5x - \cos x\right) + C$$

$$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + C$$

(22) $$\int \sin 2x \cos 4x dx = \int \cos 4x \sin 2x dx = \int \frac{1}{2}(\sin 6x - \sin 2x)dx$$

$$= \frac{1}{2}\left(-\frac{1}{6}\cos 6x + \frac{1}{2}\cos 2x\right) + C = -\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x + C$$

(23) $$\int (2\sin 5x - 3\cos 3x)^2 dx = \int (4\sin^2 5x - 12\sin 5x \cos 3x + 9\cos^2 3x)dx$$

$$= \int \left\{4 \cdot \frac{1 - \cos 10x}{2} - 12 \cdot \frac{1}{2}(\sin 8x + \sin 2x) + 9 \cdot \frac{1 + \cos 6x}{2}\right\} dx$$

$$= 2\left(x - \frac{1}{10}\sin 10x\right) - 6\left(-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right) + \frac{9}{2}\left(x + \frac{1}{6}\sin 6x\right) + C$$

$$= \frac{13}{2}x - \frac{1}{5}\sin 10x + \frac{3}{4}\cos 8x + 3\cos 2x + \frac{3}{4}\sin 6x + C$$

(24) $\displaystyle\int(2\cos 2x-\cos 5x)^2dx=\int(\cos^2 5x-4\cos 5x\cos 2x+4\cos^2 2x)dx$

$$=\int\left\{\frac{1+\cos 10x}{2}-4\cdot\frac{1}{2}(\cos 7x+\cos 3x)+4\cdot\frac{1+\cos 4x}{2}\right\}dx$$

$$=\frac{1}{2}\left(x+\frac{1}{10}\sin 10x\right)-2\left(\frac{1}{7}\sin 7x+\frac{1}{3}\sin 3x\right)+2\left(x+\frac{1}{4}\sin 4x\right)+C$$

$$=\frac{5}{2}x+\frac{1}{20}\sin 10x-\frac{2}{7}\sin 7x-\frac{2}{3}\sin 3x+\frac{1}{2}\sin 4x+C$$

(25) $\displaystyle\int(2\sin 3x-\cos 4x)^2dx=\int(\cos 4x-2\sin 3x)^2dx=\int(\cos^2 4x-4\cos 4x\sin 3x+4\sin^2 3x)dx$

$$=\int\left\{\frac{1+\cos 8x}{2}-4\cdot\frac{1}{2}(\sin 7x-\sin x)+4\cdot\frac{1-\cos 6x}{2}\right\}dx$$

$$=\int\left(\frac{5}{2}+\frac{1}{2}\cos 8x-2\sin 7x+2\sin x-2\cos 6x\right)dx$$

$$=\frac{5}{2}x+\frac{1}{16}\sin 8x+\frac{2}{7}\cos 7x-2\cos x-\frac{1}{3}\sin 6x+C$$

(26) $\displaystyle\int(2\sin 6x-3\cos 2x)^2dx=\int(4\sin^2 6x-12\sin 6x\cos 2x+9\cos^2 2x)dx$

$$=\int\left\{4\cdot\frac{1-\cos 12x}{2}-12\cdot\frac{1}{2}(\sin 8x+\sin 4x)+9\cdot\frac{1+\cos 4x}{2}\right\}dx$$

$$=\int\left(\frac{13}{2}-2\cos 12x-6\sin 8x-6\sin 4x+\frac{9}{2}\cos 4x\right)dx$$

$$=\frac{13}{2}x-\frac{1}{6}\sin 12x+\frac{3}{4}\cos 8x+\frac{3}{2}\cos 4x+\frac{9}{8}\sin 4x+C$$

(27) $\displaystyle\int(2\cos 7x-5\sin 3x)^2dx=\int(4\cos^2 7x-20\cos 7x\sin 3x+25\sin^2 3x)dx$

$$=\int\left\{4\cdot\frac{1+\cos 14x}{2}-20\cdot\frac{1}{2}(\sin 10x-\sin 4x)+25\cdot\frac{1-\cos 6x}{2}\right\}dx$$

$$=\int\left(\frac{29}{2}+2\cos 14x-10\sin 10x+10\sin 4x-\frac{25}{2}\cos 6x\right)dx$$

$$=\frac{29}{2}x+\frac{1}{7}\sin 14x+\cos 10x-\frac{5}{2}\cos 4x-\frac{25}{12}\sin 6x+C$$

(28) $\displaystyle\int(2\cos 7x-5\cos 4x)^2dx=\int(4\cos^2 7x-20\cos 7x\cos 4x+25\cos^2 4x)dx$

$$=\int\left\{4\cdot\frac{1+\cos 14x}{2}-20\cdot\frac{1}{2}(\cos 11x+\cos 3x)+25\cdot\frac{1+\cos 8x}{2}\right\}dx$$

$$=\int\left(\frac{29}{2}+2\cos 14x-10\cos 11x-10\cos 3x+\frac{25}{2}\cos 8x\right)dx$$

$$=\frac{29}{2}x+\frac{1}{7}\sin 14x-\frac{10}{11}\sin 11x-\frac{10}{3}\sin 3x+\frac{25}{16}\sin 8x+C$$

(29) $\displaystyle\int (2\sin 5x - 3\cos 2x + 1)^2 dx$

$$= \int (4\sin^2 5x + 9\cos^2 2x + 1 - 12\sin 5x\cos 2x - 6\cos 2x + 4\sin 5x)dx$$

$$= \int \left\{ 4\cdot\frac{1-\cos 10x}{2} + 9\cdot\frac{1+\cos 4x}{2} + 1 - 12\cdot\frac{1}{2}(\sin 7x + \sin 3x) - 6\cos 2x + 4\sin 5x \right\} dx$$

$$= \int \left(\frac{15}{2} - 2\cos 10x + \frac{9}{2}\cos 4x - 6\sin 7x - 6\sin 3x - 6\cos 2x + 4\sin 5x \right) dx$$

$$= \frac{15}{2}x - \frac{1}{5}\sin 10x + \frac{9}{8}\sin 4x + \frac{6}{7}\cos 7x + 2\cos 3x - 3\sin 2x - \frac{4}{5}\cos 5x + C$$