<table>
<tr><td>科目名</td><td>微積分学</td><td>対象</td><td>1OB-A</td><td>学部
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<tr><td colspan="4">平成 24 年 1 月 25 日 (水) 3 回目
(～ 時限目)</td><td>担当</td><td>石川 学</td><td>学年</td><td></td><td>氏名</td><td></td><td></td></tr>
<tr><td>試験
時間</td><td>60 分</td><td>注意
事項</td><td colspan="8">① 筆記用具以外持込不可
~~2. 下記のみ参照・持込可~~
( )</td></tr>
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平成 23 年度後期定期試験

※解答用紙の裏面使用可

$\boxed{1}$ $f(x,y)=x^3+y^3+\dfrac{9}{2}x^2+\dfrac{9}{2}y^2+6xy$ について，次の問いに答えよ．

(1) $f(x,y)$ の停留点を求めよ．

(2) $f(x,y)$ の極値を求めよ．

(3) $x^2y+xy^2=2$ に制限した $f(x,y)$ が点 $(1,1)$ で極値をとるかどうか調べよ．

$\boxed{2}$ 次の積分を求めよ．

(1) $\displaystyle\int_1^2\left(\int_1^x \frac{x}{y}dy\right)dx$

(2) $\displaystyle\int_1^{\sqrt{3}}\left(\int_x^{x^2} \frac{x}{x^2+y^2}dy\right)dx$

(3) $\displaystyle\int_0^2\left(\int_{\frac{x}{2}}^1 e^{-y^2}dy\right)dx$ （順序変更）

(4) $\displaystyle\int\int_D \frac{y^2}{x^2}dxdy \quad \left(D:1\leqq x^2+y^2\leqq 4,\ -\frac{x}{\sqrt{3}}\leqq y\leqq \sqrt{3}x\right)$

(5) $\displaystyle\int\int_D (x+y)dxdy \quad (D:x\leqq x^2+y^2\leqq 1,\ x\geqq 0,\ y\geqq 0)$

H23後・定

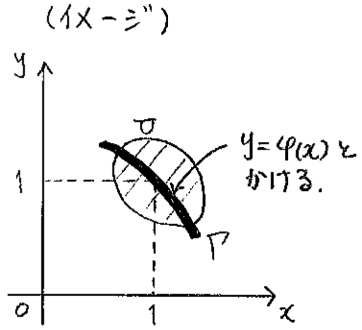
$\boxed{1}$ (3) $x^2y+xy^2=2$ に制限した $f(x,y)=x^3+y^3+\frac{9}{2}x^2+\frac{9}{2}y^2+6xy$ が点 $(1,1)$ で極値をとるかどうか調べよ。

(解) $g(x,y)=x^2y+xy^2-2$, $\Gamma : g(x,y)=0$ とおく.

$g(1,1)=0$ であり, $g_y(x,y)=x^2+2xy$ より

$g_y(1,1)=3\neq 0$ だから, 陰関数定理より

$(1,1)$ のある近傍 U 内では y は x の C^∞ 級関数 $y=\varphi(x)$ となり,



$$\varphi(1)=1 ,\quad x^2\varphi(x)+x\varphi(x)^2=2 \quad \cdots ①$$

をみたす. ここで, ① を微分すると

$$2x\cdot\varphi(x)+x^2\cdot\varphi'(x)+1\cdot\varphi(x)^2+x\cdot\{2\varphi(x)\cdot\varphi'(x)\}=0$$

$$\therefore\ 2x\varphi(x)+x^2\varphi'(x)+\varphi(x)^2+2x\varphi(x)\varphi'(x)=0 \quad \cdots ②$$

② に $x=1$ を代入して

$$2\cdot1\cdot\varphi(1)+1^2\cdot\varphi'(1)+\varphi(1)^2+2\cdot1\cdot\varphi(1)\cdot\varphi'(1)=0$$

$$2+\varphi'(1)+1+2\varphi'(1)=0 \qquad \therefore\ \varphi'(1)=-1$$

また, ② を微分すると

$$2\cdot\varphi(x)+2x\cdot\varphi'(x)+2x\cdot\varphi'(x)+x^2\cdot\varphi''(x)+2\varphi(x)\cdot\varphi'(x)+2\cdot\varphi(x)\varphi'(x)+2x\cdot\varphi'(x)\cdot\varphi'(x)+2x\varphi(x)\cdot\varphi''(x)=0$$

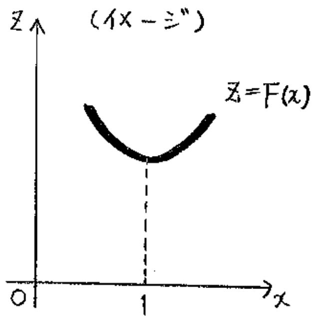
$$\therefore\ 2\varphi(x)+4x\varphi'(x)+x^2\varphi''(x)+4\varphi(x)\varphi'(x)+2x\varphi'(x)^2+2x\varphi(x)\varphi''(x)=0 \quad \cdots ③$$

③ に $x=1$ を代入して

$$2\varphi(1)+4\cdot1\cdot\varphi'(1)+1^2\cdot\varphi''(1)+4\varphi(1)\varphi'(1)+2\cdot1\cdot\varphi'(1)^2+2\cdot1\cdot\varphi(1)\varphi''(1)=0$$

$$2-4+\varphi''(1)-4+2+2\varphi''(1)=0 \qquad \therefore\ \varphi''(1)=\frac{4}{3}$$

さて, U 内では, Γ に制限した $f(x,y)$ は



$$f|_{\Gamma}(x,y)=f(x,\varphi(x))$$
$$=x^3+\varphi(x)^3+\frac{9}{2}x^2+\frac{9}{2}\varphi(x)^2+6x\varphi(x)$$

となる. これを $F(x)$ とおき, $F(x)$ が $x=1$ で極値をとるかどうか調べればよい.

$$F'(x)=3x^2+3\varphi(x)^2\cdot\varphi'(x)+9x+9\varphi(x)\cdot\varphi'(x)+6\cdot\varphi(x)+6x\cdot\varphi'(x)$$

$$F'(1)=3\cdot1^2+3\varphi(1)^2\varphi'(1)+9\cdot1+9\varphi(1)\varphi'(1)+6\varphi(1)+6\cdot1\cdot\varphi'(1)$$
$$=3-3+9-9+6-6$$
$$=0$$

$$F''(x)=6x+\{6\varphi(x)\cdot\varphi'(x)\}\cdot\varphi'(x)+3\varphi(x)^2\cdot\varphi''(x)+9+9\varphi'(x)\cdot\varphi'(x)+9\varphi(x)\cdot\varphi''(x)+6\varphi'(x)+6\cdot\varphi'(x)+6x\cdot\varphi''(x)$$
$$=6x+6\varphi(x)\varphi'(x)^2+3\varphi(x)^2\varphi''(x)+9+9\varphi'(x)^2+9\varphi(x)\varphi''(x)+12\varphi'(x)+6x\varphi''(x)$$

$$F''(1)=6\cdot1+6\varphi(1)\varphi'(1)^2+3\varphi(1)^2\varphi''(1)+9+9\varphi'(1)^2+9\varphi(1)\varphi''(1)+12\varphi'(1)+6\cdot1\cdot\varphi''(1)$$
$$=6+6+4+9+9+12-12+8$$
$$=42>0 \quad (\Longrightarrow F(x) \text{ は } x=1 \text{ の近くで下に凸})$$

よって, $F(1)=1+1+\frac{9}{2}+\frac{9}{2}+6=17$: 極小値

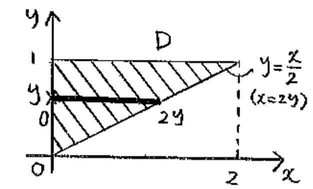
$\boxed{2}$

(1) $\displaystyle\int_1^2\left(\int_1^x\frac{x}{y}dy\right)dx=\int_1^2\left[x\log y\right]_{y=1}^{y=x}dx=\int_1^2 x\log x\,dx=\left[\frac{x^2}{2}\log x-\frac{x^2}{4}\right]_1^2=2\log2-\frac{3}{4}$

(2) $\displaystyle\int_1^{\sqrt3}\left(\int_x^{x^2}\frac{x}{x^2+y^2}dy\right)dx=\int_1^{\sqrt3}\left[\arctan\frac{y}{x}\right]_{y=x}^{y=x^2}=\int_1^{\sqrt3}\left(\arctan x-\frac{\pi}{4}\right)dx$

$\displaystyle=\left[\left\{x\arctan x-\frac{1}{2}\log(1+x^2)\right\}-\frac{\pi}{4}x\right]_1^{\sqrt3}=\frac{\sqrt3}{12}\pi-\frac{1}{2}\log2$

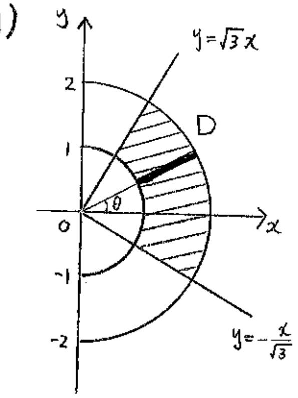
(3) $\displaystyle\int_0^2\left(\int_{\frac{x}{2}}^1 e^{-y^2}dy\right)dx$ (順序変更)



($D: 0\leqq x\leqq 2,\ \frac{x}{2}\leqq y\leqq 1 \xrightarrow{\text{図}} D: 0\leqq y\leqq 1,\ 0\leqq x\leqq 2y$)

$\displaystyle=\int_0^1\left(\int_0^{2y}e^{-y^2}dx\right)dy=\int_0^1\left[xe^{-y^2}\right]_{x=0}^{x=2y}dy=\int_0^1 2ye^{-y^2}dy=\left[-e^{-y^2}\right]_0^1=1-\frac{1}{e}$

(4) $x=r\cos\theta,\ y=r\sin\theta$ とおくと (← r:原点からの距離, θ:x軸となす角)



$\displaystyle\iint_D\frac{y^2}{x^2}dxdy \quad \left(D: 1\leqq x^2+y^2\leqq4,\ -\frac{x}{\sqrt3}\leqq y\leqq\sqrt3 x\right)$

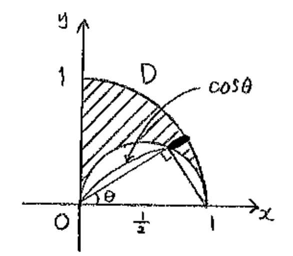
$\displaystyle=\iint_{D'}\frac{r^2\sin^2\theta}{r^2\cos^2\theta}\cdot r\,drd\theta \quad \left(D': -\frac{\pi}{6}\leqq\theta\leqq\frac{\pi}{3},\ 1\leqq r\leqq2\right)$

↑ Jacobian

$\displaystyle=\iint_{D'}r\tan^2\theta\,drd\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\int_1^2 r\tan^2\theta\,dr\right)d\theta=\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\left[\frac{r^2}{2}\tan^2\theta\right]_{r=1}^{r=2}d\theta$

$\displaystyle=\frac{3}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\frac{1}{\cos^2\theta}-1\right)d\theta=\frac{3}{2}\left[\tan\theta-\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}}=2\sqrt3-\frac{3}{4}\pi$

(5) $x=r\cos\theta,\ y=r\sin\theta$ とおくと



$\displaystyle\iint_D(x+y)dxdy \quad (D: x\leqq x^2+y^2\leqq1,\ x\geqq0,\ y\geqq0)$

$\displaystyle=\iint_{D'}(r\cos\theta+r\sin\theta)\cdot r\,drd\theta \quad \left(D': 0\leqq\theta\leqq\frac{\pi}{2},\ \cos\theta\leqq r\leqq1\right)$

$\displaystyle=\iint_{D'}r^2(\cos\theta+\sin\theta)drd\theta=\int_0^{\frac{\pi}{2}}\left\{\int_{\cos\theta}^1 r^2(\cos\theta+\sin\theta)dr\right\}d\theta$

$\displaystyle=\int_0^{\frac{\pi}{2}}\left[\frac{r^3}{3}(\cos\theta+\sin\theta)\right]_{r=\cos\theta}^{r=1}d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{3}(1-\cos^3\theta)(\cos\theta+\sin\theta)d\theta$

$\displaystyle=\frac{1}{3}\int_0^{\frac{\pi}{2}}(\cos\theta+\sin\theta-\cos^4\theta-\cos^3\theta\sin\theta)d\theta=\frac{1}{3}\left(1+1-\frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2}+\left[\frac{1}{4}\cos^4\theta\right]_0^{\frac{\pi}{2}}\right)$

$\displaystyle=\frac{7}{12}-\frac{\pi}{16}$

$x\leqq x^2+y^2$
$(x-\frac{1}{2})^2+y^2\geqq\frac{1}{4}$