1．導関数

$f(x)$ を開区間 I で微分可能な関数，すなわち，各要素 $x \in I$ に対し，極限値

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

が存在するとき，$f'(x)$ を $f(x)$ の**導関数**という．また，$f(x)$ から $f'(x)$ を求めることを，$f(x)$ を x について**微分する**という．一般に，微分可能ならば連続である．

2．四則演算に関する微分法則

$f(x), g(x)$ を開区間 I で微分可能な関数とするとき，次が成り立つ．

(1) $\{kf(x)\}' = kf'(x)$　(k は定数)

(2) $\{f(x) \pm g(x)\}' = f'(x) \pm g'(x)$　（複号同順）

(3) $\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$

(4) $\left\{ \dfrac{f(x)}{g(x)} \right\}' = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$　　特に　$\left\{ \dfrac{1}{g(x)} \right\}' = -\dfrac{g'(x)}{g(x)^2}$

ただし，$g(x) \neq 0$ とする．

3．合成関数の微分法則

$f(x), g(x)$ をそれぞれ開区間 I, J で微分可能な関数とし，$f(I) \subset J$ をみたすとする．このとき

$$\{g(f(x))\}' = g'(f(x)) \cdot f'(x)$$

が成り立つ．

4．逆三角関数

(1) $\sin : \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right] \longrightarrow [-1, 1]$ の逆関数を arcsin（アークサイン）で表す．

(2) $\cos : [0, \pi] \longrightarrow [-1, 1]$ の逆関数を arccos （アークコサイン）で表す．

(3) $\tan : \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2} \right) \longrightarrow \mathbb{R}$ の逆関数を arctan（アークタンジェント）で表す．

5．代表的な導関数

(1) $(c)' = 0$　(c は定数)

(2) $(x^n)' = nx^{n-1}$　$(n \in \mathbb{Z})$

(3) $(x^\alpha)' = \alpha x^{\alpha - 1}$　$(\alpha \neq 0)$

(4) $(\sqrt{x})' = \dfrac{1}{2\sqrt{x}}$

(5) $(e^x)' = e^x$

(6) $(\log |x|)' = \dfrac{1}{x}$

(7) $(\sin x)' = \cos x$

(8) $(\cos x)' = -\sin x$

(9) $(\tan x)' = \dfrac{1}{\cos^2 x}$

(10) $(\arcsin x)' = \dfrac{1}{\sqrt{1 - x^2}}$

(11) $(\arccos x)' = -\dfrac{1}{\sqrt{1 - x^2}}$

(12) $(\arctan x)' = \dfrac{1}{1 + x^2}$

【例題 1】

(1)
$$\left(\frac{x^2+3x+4}{x^2-5x+11}\right)' = \frac{(2x+3)\cdot(x^2-5x+11)-(x^2+3x+4)\cdot(2x-5)}{(x^2-5x+11)^2}$$
$$= \frac{-8x^2+14x+53}{(x^2-5x+11)^2}$$

(2)
$$\left(\frac{x^2-6x+2}{x^2+7x+15}\right)' = \frac{(2x-6)\cdot(x^2+7x+15)-(x^2-6x+2)\cdot(2x+7)}{(x^2+7x+15)^2}$$
$$= \frac{13x^2+26x-104}{(x^2+7x+15)^2}$$

(3)
$$\left(\arctan\sqrt{\frac{11x+7}{6x-5}}\right)'$$
$$= \frac{1}{1+\dfrac{11x+7}{6x-5}}\cdot\left\{\frac{1}{2\sqrt{\dfrac{11x+7}{6x-5}}}\cdot\frac{11\cdot(6x-5)-(11x+7)\cdot 6}{(6x-5)^2}\right\}$$
$$= \frac{6x-5}{17x+2}\cdot\frac{1}{2}\sqrt{\frac{6x-5}{11x+7}}\cdot\frac{-97}{(6x-5)^2}$$
$$= \frac{-97}{2(17x+2)(6x-5)}\sqrt{\frac{6x-5}{11x+7}}$$

(4)
$$\left(\arctan\sqrt{\frac{13x-3}{8x+5}}\right)'$$
$$= \frac{1}{1+\dfrac{13x-3}{8x+5}}\cdot\left\{\frac{1}{2\sqrt{\dfrac{13x-3}{8x+5}}}\cdot\frac{13\cdot(8x+5)-(13x-3)\cdot 8}{(8x+5)^2}\right\}$$
$$= \frac{8x+5}{21x+2}\cdot\frac{1}{2}\sqrt{\frac{8x+5}{13x-3}}\cdot\frac{89}{(8x+5)^2}$$
$$= \frac{89}{2(21x+2)(8x+5)}\sqrt{\frac{8x+5}{13x-3}}$$

(5)
$$(\arcsin\sqrt{1-7x^2})' = \frac{1}{\sqrt{1-(1-7x^2)}}\cdot\frac{-14x}{2\sqrt{1-7x^2}}$$
$$= \frac{1}{\sqrt{7x^2}}\cdot\frac{-7x}{\sqrt{1-7x^2}}$$
$$= \frac{1}{\sqrt{7}\,|x|}\cdot\frac{-7x}{\sqrt{1-7x^2}}$$
$$= -\frac{\sqrt{7}\,x}{|x|\sqrt{1-7x^2}}$$

(6)
$$\left(\arccos\frac{1}{\sqrt{3x^2+1}}\right)' = -\frac{1}{\sqrt{1-\dfrac{1}{3x^2+1}}}\cdot\left(-\frac{\dfrac{6x}{2\sqrt{3x^2+1}}}{3x^2+1}\right)$$
$$= \sqrt{\frac{3x^2+1}{3x^2}}\cdot\frac{3x}{(3x^2+1)\sqrt{3x^2+1}}$$
$$= \frac{\sqrt{3x^2+1}}{\sqrt{3}\,|x|}\cdot\frac{3x}{(3x^2+1)\sqrt{3x^2+1}}$$
$$= \frac{\sqrt{3}\,x}{|x|(3x^2+1)}$$

(7)
$$\left(\arcsin\frac{2}{\sqrt{x^2+4}}\right)' = \frac{1}{\sqrt{1-\dfrac{4}{x^2+4}}}\cdot\left(-\frac{\dfrac{2\cdot 2x}{2\sqrt{x^2+4}}}{x^2+4}\right)$$
$$= \sqrt{\frac{x^2+4}{x^2}}\cdot\frac{-2x}{(x^2+4)\sqrt{x^2+4}}$$
$$= \frac{\sqrt{x^2+4}}{|x|}\cdot\frac{-2x}{(x^2+4)\sqrt{x^2+4}}$$
$$= -\frac{2x}{|x|(x^2+4)}$$

(8)
$$\left(\frac{x^3\arctan x}{\sqrt{1+x^2}}\right)'$$
$$= \frac{\left(3x^2\cdot\arctan x + x^3\cdot\dfrac{1}{1+x^2}\right)\cdot\sqrt{1+x^2} - x^3\arctan x\cdot\dfrac{2x}{2\sqrt{1+x^2}}}{1+x^2}$$
$$= \frac{\left(3x^2\arctan x + \dfrac{x^3}{1+x^2}\right)\sqrt{1+x^2} - \dfrac{x^4\arctan x}{\sqrt{1+x^2}}}{1+x^2}$$
$$= \frac{\left(3x^2\arctan x + \dfrac{x^3}{1+x^2}\right)(1+x^2) - x^4\arctan x}{(1+x^2)\sqrt{1+x^2}}$$
$$= \frac{3x^2(1+x^2)\arctan x + x^3 - x^4\arctan x}{(1+x^2)\sqrt{1+x^2}}$$
$$= \frac{(3x^2+2x^4)\arctan x + x^3}{(1+x^2)\sqrt{1+x^2}}$$

$\boxed{1}$ 次の計算をせよ．答えだけでなく途中計算も書くこと．

(1) $\left(\dfrac{-2x^2-5x+1}{x^2+x+3}\right)'$

(2) $\left(\dfrac{x^2+3x+1}{x^2-5x+7}\right)'$

(3) $\left(\dfrac{x^2+7x-2}{x^2-5x+13}\right)'$

(4) $\left(\dfrac{x^2-2x+5}{x^2+7x+13}\right)'$

(5) $\left(\arctan\sqrt{\dfrac{10x+3}{7x-5}}\right)'$

(6) $\left(\arctan\sqrt{\dfrac{5x-2}{9x+7}}\right)'$

(7) $\left(\arctan\sqrt{\dfrac{3x-8}{2x+7}}\right)'$

(8) $\left(\arctan\sqrt{\dfrac{4x-3}{9x+5}}\right)'$

(9) $(\arccos\sqrt{1-9x^2}\,)'$

(10) $\left(\arcsin\dfrac{1}{\sqrt{4x^2+1}}\right)'$

(11) $\left(\arccos \dfrac{5}{\sqrt{x^2+25}}\right)'$

(12) $\left(\dfrac{x^4 \arctan x}{\sqrt{1+x^2}}\right)'$

(13) $\left\{ \dfrac{(\arctan x)^5}{\sqrt{1+x^2}} \right\}'$

(14) $\left(\dfrac{x^4 \arcsin x}{\sqrt{1-x^2}} \right)'$

解答

$\boxed{1}$

(1) $$\left(\frac{-2x^2-5x+1}{x^2+x+3}\right)' = \frac{(-4x-5)\cdot(x^2+x+3)-(-2x^2-5x+1)\cdot(2x+1)}{(x^2+x+3)^2}$$
$$= \frac{3x^2-14x-16}{(x^2+x+3)^2}$$

(2) $$\left(\frac{x^2+3x+1}{x^2-5x+7}\right)' = \frac{(2x+3)\cdot(x^2-5x+7)-(x^2+3x+1)\cdot(2x-5)}{(x^2-5x+7)^2}$$
$$= \frac{-8x^2+12x+26}{(x^2-5x+7)^2}$$

(3) $$\left(\frac{x^2+7x-2}{x^2-5x+13}\right)' = \frac{(2x+7)\cdot(x^2-5x+13)-(x^2+7x-2)\cdot(2x-5)}{(x^2-5x+13)^2}$$
$$= \frac{-12x^2+30x+81}{(x^2-5x+13)^2}$$

(4) $$\left(\frac{x^2-2x+5}{x^2+7x+13}\right)' = \frac{(2x-2)\cdot(x^2+7x+13)-(x^2-2x+5)\cdot(2x+7)}{(x^2+7x+13)^2}$$
$$= \frac{9x^2+16x-61}{(x^2+7x+13)^2}$$

(5) $$\left(\arctan\sqrt{\frac{10x+3}{7x-5}}\right)'$$
$$= \frac{1}{1+\dfrac{10x+3}{7x-5}}\cdot\left\{\frac{1}{2\sqrt{\dfrac{10x+3}{7x-5}}}\cdot\frac{10\cdot(7x-5)-(10x+3)\cdot 7}{(7x-5)^2}\right\}$$
$$= \frac{7x-5}{17x-2}\cdot\frac{1}{2}\sqrt{\frac{7x-5}{10x+3}}\cdot\frac{-71}{(7x-5)^2}$$
$$= \frac{-71}{2(17x-2)(7x-5)}\sqrt{\frac{7x-5}{10x+3}}$$

(6) $\left(\arctan\sqrt{\dfrac{5x-2}{9x+7}}\right)'$

$$= \frac{1}{1+\dfrac{5x-2}{9x+7}}\cdot\left\{\frac{1}{2\sqrt{\dfrac{5x-2}{9x+7}}}\cdot\frac{5\cdot(9x+7)-(5x-2)\cdot 9}{(9x+7)^2}\right\}$$

$$= \frac{9x+7}{14x+5}\cdot\frac{1}{2}\sqrt{\frac{9x+7}{5x-2}}\cdot\frac{53}{(9x+7)^2}$$

$$= \frac{53}{2(14x+5)(9x+7)}\sqrt{\frac{9x+7}{5x-2}}$$

(7) $\left(\arctan\sqrt{\dfrac{3x-8}{2x+7}}\right)'$

$$= \frac{1}{1+\dfrac{3x-8}{2x+7}}\cdot\left\{\frac{1}{2\sqrt{\dfrac{3x-8}{2x+7}}}\cdot\frac{3\cdot(2x+7)-(3x-8)\cdot 2}{(2x+7)^2}\right\}$$

$$= \frac{2x+7}{5x-1}\cdot\frac{1}{2}\sqrt{\frac{2x+7}{3x-8}}\cdot\frac{37}{(2x+7)^2}$$

$$= \frac{37}{2(5x-1)(2x+7)}\sqrt{\frac{2x+7}{3x-8}}$$

(8) $\left(\arctan\sqrt{\dfrac{4x-3}{9x+5}}\right)'$

$$= \frac{1}{1+\dfrac{4x-3}{9x+5}}\cdot\left\{\frac{1}{2\sqrt{\dfrac{4x-3}{9x+5}}}\cdot\frac{4\cdot(9x+5)-(4x-3)\cdot 9}{(9x+5)^2}\right\}$$

$$= \frac{9x+5}{13x+2}\cdot\frac{1}{2}\sqrt{\frac{9x+5}{4x-3}}\cdot\frac{47}{(9x+5)^2}$$

$$= \frac{47}{2(13x+2)(9x+5)}\sqrt{\frac{9x+5}{4x-3}}$$

(9)
$$\begin{aligned}(\arccos\sqrt{1-9x^2})' &= -\frac{1}{\sqrt{1-(1-9x^2)}}\cdot\frac{-18x}{2\sqrt{1-9x^2}}\\ &= \frac{1}{\sqrt{9x^2}}\cdot\frac{9x}{\sqrt{1-9x^2}}\\ &= \frac{1}{3|x|}\cdot\frac{9x}{\sqrt{1-9x^2}}\\ &= \frac{3x}{|x|\sqrt{1-9x^2}}\end{aligned}$$

(10)
$$
\left(\arcsin\frac{1}{\sqrt{4x^2+1}}\right)' = \frac{1}{\sqrt{1-\dfrac{1}{4x^2+1}}}\cdot\left(-\frac{\dfrac{8x}{2\sqrt{4x^2+1}}}{4x^2+1}\right)
$$
$$
= \sqrt{\frac{4x^2+1}{4x^2}}\cdot\frac{-4x}{(4x^2+1)\sqrt{4x^2+1}}
$$
$$
= \frac{\sqrt{4x^2+1}}{2|x|}\cdot\frac{-4x}{(4x^2+1)\sqrt{4x^2+1}}
$$
$$
= -\frac{2x}{|x|(4x^2+1)}
$$

(11)
$$
\left(\arccos\frac{5}{\sqrt{x^2+25}}\right)' = -\frac{1}{\sqrt{1-\dfrac{25}{x^2+25}}}\cdot\left(-\frac{\dfrac{5\cdot 2x}{2\sqrt{x^2+25}}}{x^2+25}\right)
$$
$$
= \sqrt{\frac{x^2+25}{x^2}}\cdot\frac{5x}{(x^2+25)\sqrt{x^2+25}}
$$
$$
= \frac{\sqrt{x^2+25}}{|x|}\cdot\frac{5x}{(x^2+25)\sqrt{x^2+25}}
$$
$$
= \frac{5x}{|x|(x^2+25)}
$$

(12)
$$
\left(\frac{x^4\arctan x}{\sqrt{1+x^2}}\right)'
$$
$$
= \frac{\left(4x^3\cdot\arctan x + x^4\cdot\dfrac{1}{1+x^2}\right)\cdot\sqrt{1+x^2} - x^4\arctan x\cdot\dfrac{2x}{2\sqrt{1+x^2}}}{1+x^2}
$$
$$
= \frac{\left(4x^3\arctan x + \dfrac{x^4}{1+x^2}\right)\sqrt{1+x^2} - \dfrac{x^5\arctan x}{\sqrt{1+x^2}}}{1+x^2}
$$
$$
= \frac{\left(4x^3\arctan x + \dfrac{x^4}{1+x^2}\right)(1+x^2) - x^5\arctan x}{(1+x^2)\sqrt{1+x^2}}
$$
$$
= \frac{4x^3(1+x^2)\arctan x + x^4 - x^5\arctan x}{(1+x^2)\sqrt{1+x^2}}
$$
$$
= \frac{(4x^3+3x^5)\arctan x + x^4}{(1+x^2)\sqrt{1+x^2}}
$$

$$(13)\quad \left\{\frac{(\arctan x)^5}{\sqrt{1+x^2}}\right\}' = \frac{\left\{5(\arctan x)^4\cdot\dfrac{1}{1+x^2}\right\}\cdot\sqrt{1+x^2}-(\arctan x)^5\cdot\dfrac{2x}{2\sqrt{1+x^2}}}{1+x^2}$$

$$= \frac{5(\arctan x)^4 - x(\arctan x)^5}{(1+x^2)\sqrt{1+x^2}}$$

$$(12)\quad \left(\frac{x^4\arcsin x}{\sqrt{1-x^2}}\right)'$$

$$= \frac{\left(4x^3\cdot\arcsin x + x^4\cdot\dfrac{1}{\sqrt{1-x^2}}\right)\cdot\sqrt{1-x^2} - x^4\arcsin x\cdot\dfrac{-2x}{2\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{\left(4x^3\arcsin x + \dfrac{x^4}{\sqrt{1-x^2}}\right)\sqrt{1-x^2} + \dfrac{x^5\arcsin x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{\left(4x^3\arcsin x + \dfrac{x^4}{\sqrt{1-x^2}}\right)(1-x^2) + x^5\arcsin x}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{4x^3(1-x^2)\arcsin x + x^4\sqrt{1-x^2} + x^5\arcsin x}{(1-x^2)\sqrt{1-x^2}}$$

$$= \frac{(4x^3-3x^5)\arcsin x + x^4\sqrt{1-x^2}}{(1-x^2)\sqrt{1-x^2}}$$