第４回　部分積分

1．部分積分

$f(x), g(x)$ を区間 I で微分可能な関数とする．このとき，I で

$$\{f(x)g(x)\}' = f'(x)g(x) + f(x)g'(x)$$

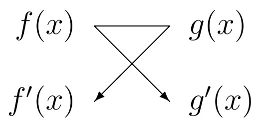
をみたすから，次が成り立つ．

$$\int \{f'(x)g(x) + f(x)g'(x)\}dx = f(x)g(x)$$

よって，$f'(x)g(x)$ の原始関数が求められるときは，

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

により $f(x)g'(x)$ の原始関数が求められる．



☆部分積分のやり方

基本的に，被積分関数が積の形のときに用いる．

(1) 被積分関数を「左上」と「右下」に配置する．

まずはやってみる．うまくいかなければ逆配置．経験をつむと，配置の仕方がわかってくる．

- ・$x^{●}$ は「左上」に置くとよさそう．(準正解)
- ・「右下」には，原始関数が（簡単に）求まるものしか置けない．(正解)　→ log は「左上」?

(2) 「左下」と「右上」を埋める．

(3) 矢印の通りに部分積分を実行する．

※特殊形

(1) 「右下」を 1 として部分積分する．

(2) 左辺と同じ積分が右辺にも現れた場合，最後は方程式を解くようにして求める．

※テクニック

「左下」をみてから「右上」を定数で修正する．

【例題 1】
次を求めよ.

(1) $\displaystyle\int xe^{-5x}dx$　　(2) $\displaystyle\int x\cos 3xdx$

(3) $\displaystyle\int x^2\sin 2xdx$　　(4) $\displaystyle\int x^3\log xdx$

(5) $\displaystyle\int \arctan xdx$　　(6) $\displaystyle\int \arcsin xdx$

(7) $\displaystyle\int 2x\arctan xdx$　　(8) $\displaystyle\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} 2x\arctan xdx$

解答

(1)
$$\begin{aligned}\int xe^{-5x}dx &= -\frac{1}{5}xe^{-5x}+\frac{1}{5}\int e^{-5x}dx\\ &= -\frac{1}{5}xe^{-5x}-\frac{1}{25}e^{-5x}\end{aligned}$$

$$\begin{array}{ccc} x & & -\dfrac{1}{5}e^{-5x} \\ 1 & & e^{-5x}\end{array}$$

(2)
$$\begin{aligned}\int x\cos 3xdx &= \frac{1}{3}x\sin 3x-\frac{1}{3}\int \sin 3xdx\\ &= \frac{1}{3}x\sin 3x+\frac{1}{9}\cos 3x\end{aligned}$$

$$\begin{array}{ccc} x & & \dfrac{1}{3}\sin 3x \\ 1 & & \cos 3x\end{array}$$

(3)
$$\begin{aligned}\int x^2\sin 2xdx &= -\frac{1}{2}x^2\cos 2x+\int x\cos 2xdx\\ &= -\frac{1}{2}x^2\cos 2x+\left(\frac{1}{2}x\sin 2x-\frac{1}{2}\int \sin 2xdx\right)\\ &= -\frac{1}{2}x^2\cos 2x+\frac{1}{2}x\sin 2x+\frac{1}{4}\cos 2x\end{aligned}$$

$$\begin{array}{ccc} x^2 & & -\dfrac{1}{2}\cos 2x \\ 2x & & \sin 2x\end{array}$$

$$\begin{array}{ccc} x & & \dfrac{1}{2}\sin 2x \\ 1 & & \cos 2x\end{array}$$

※2 回部分積分をやらなければいけないこともある.

(4)
$$\begin{aligned}\int x^3\log xdx &= \frac{1}{4}x^4\log x-\frac{1}{4}\int x^3dx\\ &= \frac{1}{4}x^4\log x-\frac{1}{16}x^4\end{aligned}$$

$$\begin{array}{ccc} \log x & & \dfrac{1}{4}x^4 \\ \dfrac{1}{x} & & x^3\end{array}$$

(5)
$$\begin{aligned}\int \arctan xdx &= x\arctan x-\int \frac{x}{1+x^2}dx\\ &= x\arctan x-\frac{1}{2}\int \frac{2x}{1+x^2}dx\\ &= x\arctan x-\frac{1}{2}\log(1+x^2)\end{aligned}$$

$$\begin{array}{ccc} \arctan x & & x \\ \dfrac{1}{1+x^2} & & 1\end{array}$$

(6)
$$\begin{aligned}\int \arcsin x dx &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx \\ &= x\arcsin x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}\cdot(-2x)dx \\ &= x\arcsin x + \frac{1}{2}\cdot 2(1-x^2)^{\frac{1}{2}} \\ &= x\arcsin x + \sqrt{1-x^2}\end{aligned}$$

$$\begin{array}{ccc} \arcsin x & & x \\ \dfrac{1}{\sqrt{1-x^2}} & & 1 \end{array}$$

(7)
$$\begin{aligned}\int 2x\arctan x dx &= (x^2+1)\arctan x - \int dx \\ &= (x^2+1)\arctan x - x\end{aligned}$$

$$\begin{array}{ccc} \arctan x & & x^2+1 \\ \dfrac{1}{1+x^2} & & 2x \end{array}$$

(8)
$$\begin{aligned}\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} 2x\arctan x dx &= \Big[(x^2+1)\arctan x - x\Big]_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} \\ &= \left\{4\cdot\arctan\sqrt{3} - \frac{4}{3}\cdot\arctan\left(-\frac{1}{\sqrt{3}}\right)\right\} \\ &\qquad - \left\{\sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right)\right\} \\ &= \left\{4\cdot\frac{\pi}{3} - \frac{4}{3}\cdot\left(-\frac{\pi}{6}\right)\right\} - \left\{\sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right)\right\} \\ &= \frac{14}{9}\pi - \frac{4\sqrt{3}}{3}\end{aligned}$$

$\boxed{1}$ 次を求めよ．

(1) $\displaystyle\int xe^{6x}dx$

(2) $\displaystyle\int xe^{-3x}dx$

(3) $\displaystyle\int x\sin 4x dx$

(4) $\displaystyle\int x\cos 7x dx$

(5) $\displaystyle\int x\log x dx$

(6) $\displaystyle\int x^4\log x dx$

(7) $\displaystyle\int \frac{\log x}{x^2} dx$

(8) $\displaystyle\int \frac{\log x}{\sqrt{x}} dx$

(9) $\int x^2 e^{-x} dx$

(10) $\int x^2 \cos x dx$

(11) $\int x(\log x)^2 dx$

(12) $\displaystyle\int_{1}^{\sqrt{3}} \arctan x dx$

(13) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}} \arcsin x dx$

(14) $\displaystyle\int_{-1}^{\sqrt{3}} 2x \arctan x dx$

(15) $\displaystyle\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} 4x^3 \arctan x dx$

解答

$\boxed{1}$

(1)
$$\begin{aligned}\int xe^{6x}dx &= \frac{1}{6}xe^{6x}-\frac{1}{6}\int e^{6x}dx \\ &= \frac{1}{6}xe^{6x}-\frac{1}{36}e^{6x}\end{aligned}$$

$$\begin{array}{cc} x & \frac{1}{6}e^{6x} \\ 1 & e^{6x} \end{array}$$

(2)
$$\begin{aligned}\int xe^{-3x}dx &= -\frac{1}{3}xe^{-3x}+\frac{1}{3}\int e^{-3x}dx \\ &= -\frac{1}{3}xe^{-3x}-\frac{1}{9}e^{-3x}\end{aligned}$$

$$\begin{array}{cc} x & -\frac{1}{3}e^{-3x} \\ 1 & e^{-3x} \end{array}$$

(3)
$$\begin{aligned}\int x\sin 4x dx &= -\frac{1}{4}x\cos 4x+\frac{1}{4}\int \cos 4x dx \\ &= -\frac{1}{4}x\cos 4x+\frac{1}{16}\sin 4x\end{aligned}$$

$$\begin{array}{cc} x & -\frac{1}{4}\cos 4x \\ 1 & \sin 4x \end{array}$$

(4)
$$\begin{aligned}\int x\cos 7x dx &= \frac{1}{7}x\sin 7x-\frac{1}{7}\int \sin 7x dx \\ &= \frac{1}{7}x\sin 7x+\frac{1}{49}\cos 7x\end{aligned}$$

$$\begin{array}{cc} x & \frac{1}{7}\sin 7x \\ 1 & \cos 7x \end{array}$$

(5)
$$\begin{aligned}\int x\log x dx &= \frac{1}{2}x^2\log x-\frac{1}{2}\int x dx \\ &= \frac{1}{2}x^2\log x-\frac{1}{4}x^2\end{aligned}$$

$$\begin{array}{cc} \log x & \frac{1}{2}x^2 \\ \frac{1}{x} & x \end{array}$$

(6)
$$\begin{aligned}\int x^4\log x dx &= \frac{1}{5}x^5\log x-\frac{1}{5}\int x^4 dx \\ &= \frac{1}{5}x^5\log x-\frac{1}{25}x^5\end{aligned}$$

$$\begin{array}{cc} \log x & \frac{1}{5}x^5 \\ \frac{1}{x} & x^4 \end{array}$$

(7)
$$\begin{aligned}\int \frac{\log x}{x^2}dx &= -\frac{\log x}{x}+\int \frac{1}{x^2}dx \\ &= -\frac{\log x}{x}-\frac{1}{x}\end{aligned}$$

$$\begin{array}{cc} \log x & -\frac{1}{x} \\ \frac{1}{x} & \frac{1}{x^2} \end{array}$$

(8)
$$\begin{aligned}\int \frac{\log x}{\sqrt{x}}dx &= 2\sqrt{x}\log x-2\int \frac{1}{\sqrt{x}}dx \\ &= 2\sqrt{x}\log x-4\sqrt{x}\end{aligned}$$

$$\begin{array}{cc} \log x & 2\sqrt{x} \\ \frac{1}{x} & \frac{1}{\sqrt{x}} \end{array}$$

(9)
$$\begin{aligned}\int x^2e^{-x}dx &= -x^2e^{-x}+2\int xe^{-x}dx\\ &= -x^2e^{-x}+2\left(-xe^{-x}+\int e^{-x}dx\right)\\ &= -x^2e^{-x}+2(-xe^{-x}-e^{-x})\\ &= -x^2e^{-x}-2xe^{-x}-2e^{-x}\end{aligned}$$

$$\begin{array}{cc} x^2 & -e^{-x} \\ 2x & e^{-x} \end{array} \qquad \begin{array}{cc} x & -e^{-x} \\ 1 & e^{-x} \end{array}$$

(10)
$$\begin{aligned}\int x^2\cos x dx &= x^2\sin x-2\int x\sin x dx\\ &= x^2\sin x-2\left(-x\cos x+\int \cos x dx\right)\\ &= x^2\sin x-2(-x\cos x+\sin x)\\ &= x^2\sin x+2x\cos x-2\sin x\end{aligned}$$

$$\begin{array}{cc} x^2 & \sin x \\ 2x & \cos x \end{array} \qquad \begin{array}{cc} x & -\cos x \\ 1 & \sin x \end{array}$$

(11)
$$\begin{aligned}\int x(\log x)^2dx &= \frac{1}{2}x^2(\log x)^2-\int x\log x dx\\ &= \frac{1}{2}x^2(\log x)^2-\left(\frac{1}{2}x^2\log x-\frac{1}{4}x^2\right)\\ &= \frac{1}{2}x^2(\log x)^2-\frac{1}{2}x^2\log x+\frac{1}{4}x^2\end{aligned}$$

$$\begin{array}{cc} (\log x)^2 & \frac{1}{2}x^2 \\ 2\log x\cdot\frac{1}{x} & x \end{array}$$

※ $\int x\log x dx$ の計算は，(5) の結果を用いた．

(12) $\displaystyle\int_1^{\sqrt{3}}\arctan x dx=\left[x\arctan x-\frac{1}{2}\log(1+x^2)\right]_1^{\sqrt{3}}$

$\displaystyle=\left(\sqrt{3}\cdot\arctan\sqrt{3}-1\cdot\arctan 1\right)-\frac{1}{2}(\log 4-\log 2)=\left(\sqrt{3}\cdot\frac{\pi}{3}-1\cdot\frac{\pi}{4}\right)-\frac{1}{2}(\log 4-\log 2)$

$\displaystyle=\frac{\pi}{\sqrt{3}}-\frac{\pi}{4}-\frac{1}{2}\log 2$

※原始関数の計算は,【例題 1】(5) の結果を用いた．

(13) $\displaystyle\int_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}\arcsin x dx=\left[x\arcsin x+\sqrt{1-x^2}\right]_{-\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$

$\displaystyle=\left\{\frac{1}{2}\cdot\arcsin\frac{1}{2}-\left(-\frac{\sqrt{3}}{2}\right)\cdot\arcsin\left(-\frac{\sqrt{3}}{2}\right)\right\}+\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)$

$\displaystyle=\left\{\frac{1}{2}\cdot\frac{\pi}{6}-\left(-\frac{\sqrt{3}}{2}\right)\cdot\left(-\frac{\pi}{3}\right)\right\}+\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)=\frac{\pi}{12}-\frac{\pi}{2\sqrt{3}}+\frac{\sqrt{3}}{2}-\frac{1}{2}$

※原始関数の計算は,【例題 1】(6) の結果を用いた．

(14) $\displaystyle\int_{-1}^{\sqrt{3}} 2x\arctan x dx = \Big[(x^2+1)\arctan x - x\Big]_{-1}^{\sqrt{3}}$

$= \{4\cdot\arctan\sqrt{3} - 2\cdot\arctan(-1)\} - \{\sqrt{3}-(-1)\} = \left\{4\cdot\dfrac{\pi}{3} - 2\cdot\left(-\dfrac{\pi}{4}\right)\right\} - \{\sqrt{3}-(-1)\}$

$= \dfrac{11}{6}\pi - \sqrt{3} - 1$

※原始関数の計算は,【例題 1】(7) の結果を用いた.

(15)
$$\begin{aligned}\int 4x^3\arctan x dx &= (x^4-1)\arctan x - \int (x^2-1)dx \\ &= (x^4-1)\arctan x - \left(\frac{1}{3}x^3 - x\right) \\ &= (x^4-1)\arctan x - \frac{1}{3}x^3 + x\end{aligned}$$

$$\begin{array}{ccc} \arctan x & & x^4-1 \\ \dfrac{1}{1+x^2} & & 4x^3 \end{array}$$

であるから

$\displaystyle\int_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}} 4x^3\arctan x dx = \left[(x^4-1)\arctan x - \frac{1}{3}x^3 + x\right]_{-\frac{1}{\sqrt{3}}}^{\sqrt{3}}$

$$= \left\{8\cdot\arctan\sqrt{3} - \left(-\frac{8}{9}\right)\cdot\arctan\left(-\frac{1}{\sqrt{3}}\right)\right\}$$
$$-\frac{1}{3}\left\{3\sqrt{3} - \left(-\frac{1}{3\sqrt{3}}\right)\right\} + \left\{\sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right)\right\}$$

$$= \left\{8\cdot\frac{\pi}{3} - \left(-\frac{8}{9}\right)\cdot\left(-\frac{\pi}{6}\right)\right\} - \frac{1}{3}\left\{3\sqrt{3} - \left(-\frac{1}{3\sqrt{3}}\right)\right\} + \left\{\sqrt{3} - \left(-\frac{1}{\sqrt{3}}\right)\right\}$$

$$= \frac{68}{27}\pi + \frac{8}{9\sqrt{3}}$$