1. 根号を含む有理形

$\sqrt{ax^2+bx+c}$ $(a>0)$ を含む有理形の積分は

$$\sqrt{ax^2+bx+c}+\sqrt{a}\,x=t$$

と置換する．このとき

$$x, \quad \frac{dx}{dt}, \quad \sqrt{ax^2+bx+c}$$

を t で表し，これらを用いる．

【例題 1】

カッコ内の置換を用いて次を求めよ．答えは t の積分を実行したところまででいいことにする．

(1) $\displaystyle\int \frac{1}{x\sqrt{x^2+1}}dx \quad \left(\sqrt{x^2+1}+x=t\right)$

(2) $\displaystyle\int \frac{2}{x\sqrt{4x^2+x+4}}dx \quad \left(\sqrt{4x^2+x+4}+2x=t\right)$

解答

(1) $\sqrt{x^2+1}=t-x$ の両辺を 2 乗すると

$x^2+1=t^2-2tx+x^2$

$2tx=t^2-1 \qquad \therefore \quad x=\dfrac{t^2-1}{2t} \ \left(=\dfrac{1}{2}\left(t-\dfrac{1}{t}\right)\right)$

また

$$\frac{dx}{dt}=\left\{\frac{1}{2}\left(t-\frac{1}{t}\right)\right\}'=\frac{1}{2}\left(1+\frac{1}{t^2}\right)=\frac{t^2+1}{2t^2}$$

さらに

$$\sqrt{x^2+1}=t-x=t-\frac{t^2-1}{2t}=\frac{t^2+1}{2t}$$

よって

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2+1}}dx &= \int \frac{1}{\dfrac{t^2-1}{2t}\cdot\dfrac{t^2+1}{2t}}\cdot\frac{t^2+1}{2t^2}dt \\ &= \int \frac{2}{t^2-1}dt \\ &= \int \frac{2}{(t-1)(t+1)}dt \\ &= \int \left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt \quad \cdots\cdots ① \\ &= \log|t-1|-\log|t+1|\end{aligned}$$

※ ① への変形は，次のどちらかでやるとよい．

(i) $\dfrac{2}{(t-1)(t+1)}=\dfrac{(t+1)-(t-1)}{(t-1)(t+1)}=\dfrac{1}{t-1}-\dfrac{1}{t+1}$

(ii) $\dfrac{2}{(t-1)(t+1)}=\dfrac{A}{t-1}+\dfrac{B}{t+1}$ と分解の形を決めてから分母をはらうと

$2=A(t+1)+B(t-1)$

$t=1$ を代入　　$2=2A \qquad \therefore \quad A=1$

$t=-1$ を代入　　$2=-2B \qquad \therefore \quad B=-1$

(2) $\sqrt{4x^2+x+4}=t-2x$ の両辺を 2 乗すると

$4x^2+x+4=t^2-4tx+4x^2$

$(4t+1)x=t^2-4 \qquad \therefore \quad x=\dfrac{t^2-4}{4t+1}$

また

$$\frac{dx}{dt}=\left(\frac{t^2-4}{4t+1}\right)'=\frac{2t\cdot(4t+1)-(t^2-4)\cdot 4}{(4t+1)^2}=\frac{2(2t^2+t+8)}{(4t+1)^2}$$

さらに

$$\sqrt{4x^2+x+4}=t-2x=t-2\cdot\frac{t^2-4}{4t+1}=\frac{2t^2+t+8}{4t+1}$$

よって

$$\begin{aligned}\int\frac{2}{x\sqrt{4x^2+x+4}}dx &= \int\frac{2}{\dfrac{t^2-4}{4t+1}\cdot\dfrac{2t^2+t+8}{4t+1}}\cdot\frac{2(2t^2+t+8)}{(4t+1)^2}dt\\ &= \int\frac{4}{t^2-4}dt\\ &= \int\frac{4}{(t-2)(t+2)}dt\\ &= \int\left(\frac{1}{t-2}-\frac{1}{t+2}\right)dt \quad \cdots\cdots②\\ &= \log|t-2|-\log|t+2|\end{aligned}$$

※ ② への変形は，次のどちらかでやるとよい．

(i) $\dfrac{4}{(t-2)(t+2)}=\dfrac{(t+2)-(t-2)}{(t-2)(t+2)}=\dfrac{1}{t-2}-\dfrac{1}{t+2}$

(ii) $\dfrac{4}{(t-2)(t+2)}=\dfrac{A}{t-2}+\dfrac{B}{t+2}$ と分解の形を決めてから分母をはらうと

$4=A(t+2)+B(t-2)$

$t=2$ を代入 $\qquad 4=4A \qquad \therefore \quad A=1$

$t=-2$ を代入 $\qquad 4=-4B \qquad \therefore \quad B=-1$

$\boxed{1}$ カッコ内の置換を用いて次を求めよ．

(1) $\displaystyle\int \frac{1}{\sqrt{x^2-13x+19}}dx \quad \left(\sqrt{x^2-13x+19}+x=t\right)$

(2) $\displaystyle\int \frac{1}{x\sqrt{x^2+5x+1}}dx \quad \left(\sqrt{x^2+5x+1}+x=t\right)$

(3) $\displaystyle\int \frac{3}{x\sqrt{4x^2-5x+9}}dx \quad \left(\sqrt{4x^2-5x+9}+2x=t\right)$

(4) $\displaystyle\int \frac{5}{x\sqrt{16x^2-7x+25}}dx \quad \left(\sqrt{16x^2-7x+25}+4x=t\right)$

(5) $\displaystyle\int \frac{2}{x\sqrt{5x^2+3x+4}}dx \quad \left(\sqrt{5x^2+3x+4}+\sqrt{5}\,x=t\right)$

(6) $\displaystyle\int \frac{1}{x\sqrt{7x^2+5x-6}}dx \quad \left(\sqrt{7x^2+5x-6}+\sqrt{7}\,x=t\right)$

(7) $\displaystyle\int \frac{2}{2\sqrt{x^2+1}+1}dx \quad \left(\sqrt{x^2+1}+x=t\right)$

(8) $\displaystyle\int \frac{1}{x^2\sqrt{x^2+1}}dx \quad \left(\sqrt{x^2+1}+x=t\right)$

解答

$\boxed{1}$

(1) $\sqrt{x^2-13x+19}=t-x$ の両辺を 2 乗すると

$x^2-13x+19=t^2-2tx+x^2$

$(2t-13)x=t^2-19 \qquad \therefore \quad x=\dfrac{t^2-19}{2t-13}$

また

$$\frac{dx}{dt}=\left(\frac{t^2-19}{2t-13}\right)'=\frac{2t\cdot(2t-13)-(t^2-19)\cdot 2}{(2t-13)^2}=\frac{2(t^2-13t+19)}{(2t-13)^2}$$

さらに

$$\sqrt{x^2-13x+19}=t-x=t-\frac{t^2-19}{2t-13}=\frac{t^2-13t+19}{2t-13}$$

よって

$$\begin{aligned}\int\frac{1}{\sqrt{x^2-13x+19}}dx &= \int\frac{1}{\dfrac{t^2-13t+19}{2t-13}}\cdot\frac{2(t^2-13t+19)}{(2t-13)^2}dt\\ &= \int\frac{2}{2t-13}dt\\ &= \log|2t-13|\end{aligned}$$

(2) $\sqrt{x^2+5x+1}=t-x$ の両辺を 2 乗すると

$x^2+5x+1=t^2-2tx+x^2$

$(2t+5)x=t^2-1 \qquad \therefore \quad x=\dfrac{t^2-1}{2t+5}$

また

$$\frac{dx}{dt}=\left(\frac{t^2-1}{2t+5}\right)'=\frac{2t\cdot(2t+5)-(t^2-1)\cdot 2}{(2t+5)^2}=\frac{2(t^2+5t+1)}{(2t+5)^2}$$

さらに

$$\sqrt{x^2+5x+1}=t-x=t-\frac{t^2-1}{2t+5}=\frac{t^2+5t+1}{2t+5}$$

よって

$$\begin{aligned}\int\frac{1}{x\sqrt{x^2+5x+1}}dx &= \int\frac{1}{\dfrac{t^2-1}{2t+5}\cdot\dfrac{t^2+5t+1}{2t+5}}\cdot\frac{2(t^2+5t+1)}{(2t+5)^2}dt\\ &= \int\frac{2}{t^2-1}dt\\ &= \int\frac{2}{(t-1)(t+1)}dt\\ &= \int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\\ &= \log|t-1|-\log|t+1|\end{aligned}$$

(3) $\sqrt{4x^2-5x+9}=t-2x$ の両辺を 2 乗すると

$4x^2-5x+9=t^2-4tx+4x^2$

$(4t-5)x=t^2-9 \qquad \therefore \quad x=\dfrac{t^2-9}{4t-5}$

また

$$\frac{dx}{dt}=\left(\frac{t^2-9}{4t-5}\right)'=\frac{2t\cdot(4t-5)-(t^2-9)\cdot 4}{(4t-5)^2}=\frac{2(2t^2-5t+18)}{(4t-5)^2}$$

さらに

$$\sqrt{4x^2-5x+9}=t-2x=t-2\cdot\frac{t^2-9}{4t-5}=\frac{2t^2-5t+18}{4t-5}$$

よって

$$\begin{aligned}
\int\frac{3}{x\sqrt{4x^2-5x+9}}dx &= \int\frac{3}{\dfrac{t^2-9}{4t-5}\cdot\dfrac{2t^2-5t+18}{4t-5}}\cdot\frac{2(2t^2-5t+18)}{(4t-5)^2}dt\\
&= \int\frac{6}{t^2-9}dt\\
&= \int\frac{6}{(t-3)(t+3)}dt\\
&= \int\left(\frac{1}{t-3}-\frac{1}{t+3}\right)dt\\
&= \log|t-3|-\log|t+3|
\end{aligned}$$

(4) $\sqrt{16x^2-7x+25}=t-4x$ の両辺を 2 乗すると

$16x^2-7x+25=t^2-8tx+16x^2$

$(8t-7)x=t^2-25 \qquad \therefore \quad x=\dfrac{t^2-25}{8t-7}$

また

$$\frac{dx}{dt}=\left(\frac{t^2-25}{8t-7}\right)'=\frac{2t\cdot(8t-7)-(t^2-25)\cdot 8}{(8t-7)^2}=\frac{2(4t^2-7t+100)}{(8t-7)^2}$$

さらに

$$\sqrt{16x^2-7x+25}=t-4x=t-4\cdot\frac{t^2-25}{8t-7}=\frac{4t^2-7t+100}{8t-7}$$

よって

$$\begin{aligned}\int \frac{5}{x\sqrt{16x^2-7x+25}}dx &= \int \frac{5}{\dfrac{t^2-25}{8t-7}\cdot\dfrac{4t^2-7t+100}{8t-7}}\cdot\frac{2(4t^2-7t+100)}{(8t-7)^2}dt \\ &= \int \frac{10}{t^2-25}dt \\ &= \int \frac{10}{(t-5)(t+5)}dt \\ &= \int\left(\frac{1}{t-5}-\frac{1}{t+5}\right)dt \\ &= \log|t-5|-\log|t+5|\end{aligned}$$

(5) $\sqrt{5x^2+3x+4}=t-\sqrt{5}\,x$ の両辺を 2 乗すると

$5x^2+3x+4=t^2-2\sqrt{5}\,tx+5x^2$

$(2\sqrt{5}\,t+3)x=t^2-4 \qquad \therefore \quad x=\dfrac{t^2-4}{2\sqrt{5}\,t+3}$

また

$$\frac{dx}{dt}=\left(\frac{t^2-4}{2\sqrt{5}\,t+3}\right)'=\frac{2t\cdot(2\sqrt{5}\,t+3)-(t^2-4)\cdot 2\sqrt{5}}{(2\sqrt{5}\,t+3)^2}=\frac{2(\sqrt{5}\,t^2+3t+4\sqrt{5})}{(2\sqrt{5}\,t+3)^2}$$

さらに

$$\sqrt{5x^2+3x+4}=t-\sqrt{5}\,x=t-\sqrt{5}\cdot\frac{t^2-4}{2\sqrt{5}\,t+3}=\frac{\sqrt{5}\,t^2+3t+4\sqrt{5}}{2\sqrt{5}\,t+3}$$

よって

$$\begin{aligned}\int \frac{2}{x\sqrt{5x^2+3x+4}}dx &= \int \frac{2}{\dfrac{t^2-4}{2\sqrt{5}\,t+3}\cdot\dfrac{\sqrt{5}\,t^2+3t+4\sqrt{5}}{2\sqrt{5}\,t+3}}\cdot\frac{2(\sqrt{5}\,t^2+3t+4\sqrt{5})}{(2\sqrt{5}\,t+3)^2}dt \\ &= \int \frac{4}{t^2-4}dt \\ &= \int \frac{4}{(t-2)(t+2)}dt \\ &= \int\left(\frac{1}{t-2}-\frac{1}{t+2}\right)dt \\ &= \log|t-2|-\log|t+2|\end{aligned}$$

(6) $\sqrt{7x^2+5x-6}=t-\sqrt{7}\,x$ の両辺を 2 乗すると

$7x^2+5x-6=t^2-2\sqrt{7}\,tx+7x^2$

$(2\sqrt{7}\,t+5)x=t^2+6 \qquad \therefore \quad x=\dfrac{t^2+6}{2\sqrt{7}\,t+5}$

また

$$\frac{dx}{dt}=\left(\frac{t^2+6}{2\sqrt{7}\,t+5}\right)'=\frac{2t\cdot(2\sqrt{7}\,t+5)-(t^2+6)\cdot 2\sqrt{7}}{(2\sqrt{7}\,t+5)^2}=\frac{2(\sqrt{7}\,t^2+5t-6\sqrt{7})}{(2\sqrt{7}\,t+5)^2}$$

さらに

$$\sqrt{7x^2+5x-6}=t-\sqrt{7}x=t-\sqrt{7}\cdot\frac{t^2+6}{2\sqrt{7}t+5}=\frac{\sqrt{7}t^2+5t-6\sqrt{7}}{2\sqrt{7}t+5}$$

よって

$$\begin{aligned}\int\frac{1}{x\sqrt{7x^2+5x-6}}dx &= \int\frac{1}{\dfrac{t^2+6}{2\sqrt{7}t+5}\cdot\dfrac{\sqrt{7}t^2+5t-6\sqrt{7}}{2\sqrt{7}t+5}}\cdot\frac{2(\sqrt{7}t^2+5t-6\sqrt{7})}{(2\sqrt{7}t+5)^2}dt\\ &= \int\frac{2}{t^2+6}dt\\ &= \int\frac{2}{(\sqrt{6})^2+t^2}dt\\ &= \frac{2}{\sqrt{6}}\arctan\frac{t}{\sqrt{6}}\end{aligned}$$

(7)【例題 1】(1) の過程より，$x=\dfrac{t^2-1}{2t}$, $\dfrac{dx}{dt}=\dfrac{t^2+1}{2t^2}$, $\sqrt{x^2+1}=\dfrac{t^2+1}{2t}$ であるから

$$\begin{aligned}\int\frac{2}{2\sqrt{x^2+1}+1}dx &= \int\frac{2}{2\cdot\dfrac{t^2-1}{2t}+1}\cdot\frac{t^2+1}{2t^2}dt\\ &= \int\frac{t^2+1}{t(t^2+t+1)}dt\\ &= \int\left(\frac{1}{t}-\frac{1}{t^2+t+1}\right)dt\\ &= \int\left\{\frac{1}{t}-\frac{1}{(\frac{\sqrt{3}}{2})^2+(t+\frac{1}{2})^2}\right\}dt\\ &= \log|t|-\frac{1}{\frac{\sqrt{3}}{2}}\arctan\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\\ &= \log|t|-\frac{2}{\sqrt{3}}\arctan\frac{2t+1}{\sqrt{3}}\end{aligned}$$

※ $\dfrac{t^2+1}{t(t^2+t+1)}=\dfrac{A}{t}+\dfrac{Bt+C}{t^2+t+1}$

と分解の形を決めてから分母をはらうと

$t^2+1=A(t^2+t+1)+(Bt+C)t$

$t^2+1=(A+B)t^2+(A+C)t+A$

係数を比較すると

$$\begin{cases}A+B=1\\ A+C=0\\ A=1\end{cases}\qquad\therefore\quad A=1,\ B=0,\ C=-1$$

(8)【例題 1】(1) の過程より，$x = \dfrac{t^2-1}{2t}$，$\dfrac{dx}{dt} = \dfrac{t^2+1}{2t^2}$，$\sqrt{x^2+1} = \dfrac{t^2+1}{2t}$ であるから

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+1}}dx &= \int \frac{1}{\left(\dfrac{t^2-1}{2t}\right)^2 \cdot \dfrac{t^2+1}{2t}} \cdot \frac{t^2+1}{2t^2}dt \\
&= \int \frac{4t}{(t^2-1)^2}dt \\
&= 2\int (t^2-1)^{-2} \cdot 2tdt \\
&= 2 \cdot \left\{-(t^2-1)^{-1}\right\} \\
&= -\frac{2}{t^2-1}
\end{aligned}$$