**1．2 変数関数の重積分（長方形）**

$I : a \leqq x \leqq b,\ c \leqq y \leqq d$（座標軸に平行な長方形），$f(x,y) : I$ で有界とする．

$[a,b],\ [c,d]$ をそれぞれ

$$a = x_0 < x_1 < \cdots < x_m = b,\ c = y_0 < y_1 < \cdots < y_n = d$$

と分割し

$$I_{ij} : x_{i-1} \leqq x \leqq x_i,\ y_{j-1} \leqq y \leqq y_j \quad (1 \leqq i \leqq m,\ 1 \leqq j \leqq n) \qquad \text{（小長方形）}$$

とおくと

$$\triangle = \{I_{ij} \mid 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$$

は $I$ の分割になる．また

$$|\,\triangle\,| = \max\{\underbrace{\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2}}_{\text{（}I_{ij}\text{ の対角線の長さ）}} \mid 1 \leqq i \leqq m,\ 1 \leqq j \leqq n\}$$

とおき，$1 \leqq i \leqq m,\ 1 \leqq j \leqq n$ に対して

$$(\xi_i, \eta_j) \in I_{ij} \qquad \text{（小長方形の代表点）}$$

を任意にとる．

分割の仕方や代表点の取り方によらず一定の極限値

$$\lim_{|\triangle| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_i, \eta_j) \underbrace{(x_i - x_{i-1})(y_j - y_{j-1})}_{\text{（}I_{ij}\text{ の面積）}} \quad \cdots\cdots(*)$$

が存在するとき，$f(x,y)$ は $I$ で**積分可能**であるという．また，$(*)$ を $f(x,y)$ の $I$ における**重積分**といい，$\displaystyle\int\int_I f(x,y)dxdy$ で表す．

$$\int\int_I f(x,y)dxdy = \lim_{|\triangle| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\xi_i, \eta_j)(x_i - x_{i-1})(y_j - y_{j-1})$$

<u>定理</u>

$I : a \leqq x \leqq b,\ c \leqq y \leqq d$ とする．このとき，次が成り立つ．

(1) $f(x,y) : I$ で連続　$\Longrightarrow$　$f(x,y) : I$ で積分可能

(2) $f(x,y) : I$ で有界，面積 0 の集合を除いた所で連続　$\Longrightarrow$　$f(x,y) : I$ で積分可能

**2．2 変数関数の重積分（有界閉領域）**

$D : \mathbb{R}^2$ の有界閉領域，$f(x,y) : D$ で有界とする．
$D$ を含む長方形 $I : a \leqq x \leqq b,\ c \leqq y \leqq d$ を 1 つとり

$$\widetilde{f}(x,y) = \begin{cases} f(x,y) & ((x,y) \in D) \\ 0 & ((x,y) \in I \setminus D) \end{cases}$$

とおく．$\widetilde{f}(x,y)$ が $I$ で積分可能なとき，$f(x,y)$ は $D$ で**積分可能**であるといい，$f(x,y)$ の $D$ における**重積分**を

$$\int\int_D f(x,y)dxdy \int\int_I \widetilde{f}(x,y)dxdy$$

で定める．

<u>定理</u>
$D : \mathbb{R}^2$ の有界閉領域とする．このとき，次が成り立つ．

$f(x,y) : D$ で連続　$\Longrightarrow$　$f(x,y) : D$ で積分可能

**3．重積分の累次積分への変更**

$g_1(x), g_2(x) : [a,b]$ で連続，$a \leqq x \leqq b$ で $g_1(x) \leqq g_2(x)$
$D : a \leqq x \leqq b,\ g_1(x) \leqq y \leqq g_2(x)$　（縦線型領域）
$f(x,y) : D$ で連続

$$\Longrightarrow \int\int_D f(x,y)dxdy = \int_a^b \left\{ \int_{g_1(x)}^{g_2(x)} f(x,y)dy \right\} dx$$

※右辺のように順番に積分する形を**累次積分**という．

【例題 1】

次の累次積分を求めよ．

(1) $\displaystyle\int_0^2 \left(\int_0^{2-x} xy^2 dy\right) dx$　　(2) $\displaystyle\int_{-2}^1 \left\{\int_{x^2}^{2-x} (x+y) dy\right\} dx$

(3) $\displaystyle\int_2^6 \left(\int_1^{x^2} \frac{x}{y^2} dy\right) dx$　　(4) $\displaystyle\int_1^2 \left(\int_{\frac{1}{x}}^{x} xy dy\right) dx$

(5) $\displaystyle\int_1^3 \left(\int_1^{x} \log\frac{x}{y} dy\right) dx$　　(6) $\displaystyle\int_0^1 \left(\int_x^{\sqrt{3}} \frac{1}{1+y^2} dy\right) dx$

解答

(1)
$$\begin{aligned}
\int_0^2 \left(\int_0^{2-x} xy^2 dy\right) dx &= \int_0^2 \left[\frac{x}{3}y^3\right]_{y=0}^{y=2-x} dx \\
&= \int_0^3 \frac{x}{3}(2-x)^3 dx \\
&= \int_0^2 \left(\frac{8}{3}x - 4x^2 + 2x^3 - \frac{1}{3}x^4\right) dx \\
&= \left[\frac{4}{3}x^2 - \frac{4}{3}x^3 + \frac{1}{2}x^4 - \frac{1}{15}x^5\right]_0^2 \\
&= \frac{16}{3} - \frac{32}{3} + 8 - \frac{32}{15} \\
&= \frac{8}{15}
\end{aligned}$$

(2)
$$\begin{aligned}
\int_{-2}^1 \left\{\int_{x^2}^{2-x} (x+y) dy\right\} dx &= \int_{-2}^1 \left[xy + \frac{1}{2}y^2\right]_{y=x^2}^{y=2-x} dx \\
&= \int_{-2}^1 \left\{x(2-x) + \frac{1}{2}(2-x)^2 - \left(x^3 + \frac{1}{2}x^4\right)\right\} dx \\
&= \int_{-2}^1 \left(-\frac{1}{2}x^4 - x^3 - \frac{1}{2}x^2 + 2\right) dx \\
&= \left[-\frac{1}{10}x^5 - \frac{1}{4}x^4 - \frac{1}{6}x^3 + 2x\right]_{-2}^1 \\
&= \left(-\frac{1}{10} - \frac{1}{4} - \frac{1}{6} + 2\right) - \left(\frac{16}{5} - 4 + \frac{4}{3} - 4\right) \\
&= \frac{99}{20}
\end{aligned}$$

(3) $$\int_2^6 \left( \int_1^{x^2} \frac{x}{y^2} dy \right) dx = \int_2^6 \left[ -\frac{x}{y} \right]_{y=1}^{y=x^2} dx$$
$$= \int_2^6 \left\{ \left( -\frac{1}{x} \right) - (-x) \right\} dx$$
$$= \int_2^6 \left( x - \frac{1}{x} \right) dx$$
$$= \left[ \frac{1}{2} x^2 - \log x \right]_2^6$$
$$= \frac{1}{2}(36 - 4) - (\log 6 - \log 2)$$
$$= 16 - \log 3$$

(4) $$\int_1^2 \left( \int_{\frac{1}{x}}^{x} xy dy \right) dx = \int_1^2 \left[ \frac{x}{2} y^2 \right]_{y=\frac{1}{x}}^{y=x} dx$$
$$= \int_1^2 \left( \frac{1}{2} x^3 - \frac{1}{2x} \right) dx$$
$$= \left[ \frac{1}{8} x^4 - \frac{1}{2} \log x \right]_1^2$$
$$= \frac{1}{8}(16 - 1) - \frac{1}{2}(\log 2 - 0)$$
$$= \frac{15}{8} - \frac{1}{2} \log 2$$

(5) $$\int_1^3 \left( \int_1^x \log \frac{x}{y} dy \right) dx = \int_1^3 \left\{ \int_1^x (\log x - \log y) dy \right\} dx$$
$$= \int_1^3 \Big[ y \log x - (y \log y - y) \Big]_{y=1}^{y=x} dx$$
$$= \int_1^3 \big\{ (x-1) \log x - (x \log x - 1 \cdot 0) + (x - 1) \big\} dx$$
$$= \int_1^3 (x - 1 - \log x) dx$$
$$= \left[ \frac{1}{2}(x-1)^2 - (x \log x - x) \right]_1^3$$
$$= \frac{1}{2}(4 - 0) - (3 \log 3 - 1 \cdot 0) + (3 - 1)$$
$$= 4 - 3 \log 3$$

(6)
$$\begin{aligned}
\int_0^1 \left( \int_x^{\sqrt{3}} \frac{1}{1+y^2} dy \right) dx &= \int_0^1 \Big[ \arctan y \Big]_{y=x}^{y=\sqrt{3}} dx \\
&= \int_0^1 \left( \frac{\pi}{3} - \arctan x \right) dx \\
&= \left[ \frac{\pi}{3} x - \left\{ x \arctan x - \frac{1}{2} \log(1+x^2) \right\} \right]_0^1 \\
&= \frac{\pi}{3} - \left( 1 \cdot \frac{\pi}{4} - 0 \cdot 0 \right) + \frac{1}{2}(\log 2 - 0) \\
&= \frac{\pi}{12} + \frac{1}{2} \log 2
\end{aligned}$$

※ $\arctan x$ の原始関数は，第４回【例題 1】(5) の結果を用いた．

$\boxed{1}$ 次の累次積分を求めよ．

(1) $\displaystyle\int_0^1 \left(\int_0^{4-x} x dy\right) dx$

(2) $\displaystyle\int_0^1 \left(\int_0^{1-x} xy^2 dy\right) dx$

(3) $\displaystyle\int_0^1 \left(\int_x^2 x^2 y dy\right) dx$

(4) $\displaystyle\int_0^1 \left\{ \int_0^{x^2} (x+y)dy \right\} dx$

(5) $\displaystyle\int_0^1 \left\{ \int_0^{1-x} (5x-3y^2)dy \right\} dx$

(6) $\displaystyle\int_{-2}^{1}\left\{\int_{-1}^{x^2}(3x+7y)dy\right\}dx$

(7) $\displaystyle\int_{-1}^{2}\left\{\int_{-3}^{x^2}(3x+2y)dy\right\}dx$

(8) $\displaystyle\int_{-2}^{1}\left\{\int_{x-1}^{x^2}(7y-x)dy\right\}dx$

(9) $\displaystyle\int_{-1}^{2}\left\{\int_{2x-1}^{x^2}(2y-3x)dy\right\}dx$

(10) $\displaystyle\int_1^2 \left( \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2} dy \right) dx$

(11) $\displaystyle\int_3^6 \left( \int_1^{x^2} \frac{x}{y^2} dy \right) dx$

(12) $\displaystyle\int_1^3 \left( \int_{x^4}^{x^6} \frac{x^3}{y^2} dy \right) dx$

(13) $\displaystyle\int_1^2 \left( \int_1^x \frac{x}{y} dy \right) dx$

(14) $\displaystyle\int_2^8 \left( \int_2^x \frac{1}{x^2 y} dy \right) dx$

解答

$\boxed{1}$

(1) $\displaystyle\int_0^1 \left(\int_0^{4-x} xdy\right) dx = \int_0^1 \Big[xy\Big]_{y=0}^{y=4-x} dx = \int_0^1 x(4-x)dx = \int_0^1 (4x-x^2)dx = \left[2x^2 - \frac{1}{3}x^3\right]_0^1$

$\displaystyle = 2 - \frac{1}{3} = \frac{5}{3}$

(2) $\displaystyle\int_0^1 \left(\int_0^{1-x} xy^2dy\right) dx = \int_0^1 \left[\frac{x}{3}y^3\right]_{y=0}^{y=1-x} dx = \int_0^1 \frac{x}{3}(1-x)^3dx$

$\displaystyle = \frac{1}{3}\int_0^1 (x-3x^2+3x^3-x^4)dx = \frac{1}{3}\left[\frac{1}{2}x^2 - x^3 + \frac{3}{4}x^4 - \frac{1}{5}x^5\right]_0^1 = \frac{1}{3}\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right)$

$\displaystyle = \frac{1}{60}$

(3) $\displaystyle\int_0^1 \left(\int_x^2 x^2ydy\right) dx = \int_0^1 \left[\frac{x^2}{2}y^2\right]_{y=x}^{y=2} dx = \int_0^1 \frac{x^2}{2}(4-x^2)dx = \int_0^1 \left(2x^2 - \frac{1}{2}x^4\right) dx$

$\displaystyle = \left[\frac{2}{3}x^3 - \frac{1}{10}x^5\right]_0^1 = \frac{2}{3} - \frac{1}{10} = \frac{17}{30}$

(4) $\displaystyle\int_0^1 \left\{\int_0^{x^2} (x+y)dy\right\} dx = \int_0^1 \left[xy + \frac{1}{2}y^2\right]_{y=0}^{y=x^2} dx = \int_0^1 \left(x^3 + \frac{1}{2}x^4\right) dx$

$\displaystyle = \left[\frac{1}{4}x^4 + \frac{1}{10}x^5\right]_0^1 = \frac{1}{4} + \frac{1}{10} = \frac{7}{20}$

(5) $\displaystyle\int_0^1 \left\{\int_0^{1-x} (5x-3y^2)dy\right\} dx = \int_0^1 \Big[5xy - y^3\Big]_{y=0}^{y=1-x} dx = \int_0^1 \big\{5x(1-x) - (1-x)^3\big\}dx$

$\displaystyle = \int_0^1 (x^3 - 8x^2 + 8x - 1)dx = \left[\frac{1}{4}x^4 - \frac{8}{3}x^3 + 4x^2 - x\right]_0^1 = \frac{1}{4} - \frac{8}{3} + 4 - 1 = \frac{7}{12}$

(6) $\displaystyle\int_{-2}^1 \left\{\int_{-1}^{x^2} (3x+7y)dy\right\} dx = \int_{-2}^1 \left[3xy + \frac{7}{2}y^2\right]_{y=-1}^{y=x^2} dx$

$\displaystyle = \int_{-2}^1 \left\{\left(3x^3 + \frac{7}{2}x^4\right) - \left(-3x + \frac{7}{2}\right)\right\} dx = \int_{-2}^1 \left(\frac{7}{2}x^4 + 3x^3 + 3x - \frac{7}{2}\right) dx$

$\displaystyle = \left[\frac{7}{10}x^5 + \frac{3}{4}x^4 + \frac{3}{2}x^2 - \frac{7}{2}x\right]_{-2}^1 = \left(\frac{7}{10} + \frac{3}{4} + \frac{3}{2} - \frac{7}{2}\right) - \left(-\frac{112}{5} + 12 + 6 + 7\right) = -\frac{63}{20}$

(7) $\displaystyle\int_{-1}^2 \left\{\int_{-3}^{x^2} (3x+2y)dy\right\} dx = \int_{-1}^2 \Big[3xy + y^2\Big]_{y=-3}^{y=x^2} dx = \int_{-1}^2 \big\{(3x^3 + x^4) - (-9x+9)\big\}dx$

$\displaystyle = \int_{-1}^2 (x^4 + 3x^3 + 9x - 9)dx = \left[\frac{1}{5}x^5 + \frac{3}{4}x^4 + \frac{9}{2}x^2 - 9x\right]_{-1}^2$

$\displaystyle = \left(\frac{32}{5} + 12 + 18 - 18\right) - \left(-\frac{1}{5} + \frac{3}{4} + \frac{9}{2} + 9\right) = \frac{87}{20}$

(8) $\displaystyle\int_{-2}^{1}\left\{\int_{x-1}^{x^2}(7y-x)dy\right\}dx=\int_{-2}^{1}\left[\frac{7}{2}y^2-xy\right]_{y=x-1}^{y=x^2}dx$

$\displaystyle=\int_{-2}^{1}\left[\left(\frac{7}{2}x^4-x^3\right)-\left\{\frac{7}{2}(x-1)^2-x(x-1)\right\}\right]dx$

$\displaystyle=\int_{-2}^{1}\left\{\frac{7}{2}x^4-x^3-\frac{7}{2}(x-1)^2+x^2-x\right\}dx=\left[\frac{7}{10}x^5-\frac{1}{4}x^4-\frac{7}{6}(x-1)^3+\frac{1}{3}x^3-\frac{1}{2}x^2\right]_{-2}^{1}$

$\displaystyle=\left(\frac{7}{10}-\frac{1}{4}-0+\frac{1}{3}-\frac{1}{2}\right)-\left(-\frac{112}{5}-4+\frac{63}{2}-\frac{8}{3}-2\right)=-\frac{3}{20}$

(9) $\displaystyle\int_{-1}^{2}\left\{\int_{2x-1}^{x^2}(2y-3x)dy\right\}dx=\int_{-1}^{2}\Big[y^2-3xy\Big]_{y=2x-1}^{y=x^2}dx$

$\displaystyle=\int_{-1}^{2}\Big[(x^4-3x^3)-\big\{(2x-1)^2-3x(2x-1)\big\}\Big]dx=\int_{-1}^{2}\big\{x^4-3x^3-(2x-1)^2+6x^2-3x\big\}dx$

$\displaystyle=\left[\frac{1}{5}x^5-\frac{3}{4}x^4-\frac{1}{6}(2x-1)^3+2x^3-\frac{3}{2}x^2\right]_{-1}^{2}$

$\displaystyle=\left(\frac{32}{3}-12-\frac{9}{2}+16-6\right)-\left(-\frac{1}{5}-\frac{3}{4}+\frac{9}{2}-2-\frac{3}{2}\right)=-\frac{3}{20}$

(10) $\displaystyle\int_{1}^{2}\left(\int_{\frac{1}{x}}^{x}\frac{x^2}{y^2}dy\right)dx=\int_{1}^{2}\left[-\frac{x^2}{y}\right]_{y=\frac{1}{x}}^{y=x}dx=\int_{1}^{2}\big\{(-x)-(-x^3)\big\}dx=\int_{1}^{2}(x^3-x)dx$

$\displaystyle=\left[\frac{1}{4}x^4-\frac{1}{2}x^2\right]_{1}^{2}=(4-2)-\left(\frac{1}{4}-\frac{1}{2}\right)=\frac{9}{4}$

(11) $\displaystyle\int_{3}^{6}\left(\int_{1}^{x^2}\frac{x}{y^2}dy\right)dx=\int_{3}^{6}\left[-\frac{x}{y}\right]_{y=1}^{y=x^2}dx=\int_{3}^{6}\left\{\left(-\frac{1}{x}\right)-(-x)\right\}dx=\int_{3}^{6}\left(x-\frac{1}{x}\right)dx$

$\displaystyle=\left[\frac{1}{2}x^2-\log x\right]_{3}^{6}=\frac{1}{2}(36-9)-(\log 6-\log 3)=\frac{27}{2}-\log 2$

(12) $\displaystyle\int_{1}^{3}\left(\int_{x^4}^{x^6}\frac{x^3}{y^2}dy\right)dx=\int_{1}^{3}\left[-\frac{x^3}{y}\right]_{y=x^4}^{y=x^6}dx=\int_{1}^{3}\left\{\left(-\frac{1}{x^3}\right)-\left(-\frac{1}{x}\right)\right\}dx$

$\displaystyle=\int_{1}^{3}\left(\frac{1}{x}-\frac{1}{x^3}\right)dx=\left[\log x+\frac{1}{2x^2}\right]_{1}^{3}=(\log 3-0)+\frac{1}{2}\left(\frac{1}{9}-1\right)=\log 3-\frac{4}{9}$

(13) $\displaystyle\int_{1}^{2}\left(\int_{1}^{x}\frac{x}{y}dy\right)dx=\int_{1}^{2}\Big[x\log y\Big]_{y=1}^{y=x}dx=\int_{1}^{2}x\log x dx=\left[\frac{1}{2}x^2\log x-\frac{1}{4}x^2\right]_{1}^{2}$

$\displaystyle=\frac{1}{2}(4\log 2-1\cdot 0)-\frac{1}{4}(4-1)=2\log 2-\frac{3}{4}$

(14) $\displaystyle\int_2^8 \left(\int_2^x \frac{1}{x^2y}dy\right)dx = \int_2^8 \left[\frac{\log y}{x^2}\right]_{y=2}^{y=x} dx = \int_2^8 \left(\frac{\log x}{x^2} - \frac{\log 2}{x^2}\right)dx$

$\displaystyle= \left[\left(-\frac{\log x}{x} - \frac{1}{x}\right) + \frac{\log 2}{x}\right]_2^8 = -\left(\frac{1}{8}\log 8 - \frac{1}{2}\log 2\right) - \left(\frac{1}{8} - \frac{1}{2}\right) + \left(\frac{1}{8} - \frac{1}{2}\right)\log 2$

$\displaystyle= \frac{3}{8} - \frac{1}{4}\log 2$