$\boxed{1}$ 次の累次積分を求めよ．

(1) $\displaystyle\int_{-1}^{3}\left(\int_{x^2}^{2x+3} xy\,dy\right)dx$

(2) $\displaystyle\int_{0}^{1}\left\{\int_{x^2+1}^{x+1}(x^2+2y)\,dy\right\}dx$

(3) $\displaystyle\int_{-2}^{1}\left\{\int_{x^2}^{2-x}(x+y)dy\right\}dx$

(4) $\displaystyle\int_{0}^{1}\left(\int_{\sqrt{1-x^2}}^{x+2}xydy\right)dx$

(5) $\displaystyle\int_{\sqrt{e}}^{e^2}\left(\int_1^x \frac{1}{xy}dy\right)dx$

(6) $\displaystyle\int_0^{\frac{\pi}{3}}\left(\int_0^{\sin x} xdy\right)dx$

(7) $\displaystyle\int_0^1 \left\{ \int_0^x (x+2)e^{2y} dy \right\} dx$

(8) $\displaystyle\int_0^1 \left(\int_{-x}^{2x} x e^{y} dy \right) dx$

(9) $\displaystyle\int_1^6 \left(\int_1^x \log y dy\right) dx$

(10) $\displaystyle\int_1^3 \left(\int_{x^2}^{x^3} \frac{x}{y} dy\right) dx$

(11) $\displaystyle\int_1^2 \left(\int_1^x \log \frac{x}{y^2} dy \right) dx$

(12) $\displaystyle\int_1^2 \left(\int_1^x \frac{x^2}{y} dy \right) dx$

(13) $\displaystyle\int_0^2 \left(\int_0^{x^2} \frac{x}{1+y}\,dy \right) dx$

(14) $\displaystyle\int_1^{\sqrt{3}} \left(\int_x^{x^2} \frac{x}{x^2+y^2}\,dy \right) dx$

解答

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(1) $\displaystyle \int_{-1}^{3}\left(\int_{x^2}^{2x+3} xydy\right)dx = \int_{-1}^{3}\left[\frac{x}{2}y^2\right]_{y=x^2}^{y=2x+3}dx = \int_{-1}^{3}\frac{x}{2}\left\{(2x+3)^2 - x^4\right\}dx$

$\displaystyle = \int_{-1}^{3}\frac{1}{2}(-x^5+4x^3+12x^2+9x)dx = \frac{1}{2}\left[-\frac{1}{6}x^6+x^4+4x^3+\frac{9}{2}x^2\right]_{-1}^{3}$

$\displaystyle = \frac{1}{2}\left\{\left(-\frac{243}{2}+81+108+\frac{81}{2}\right)-\left(-\frac{1}{6}+1-4+\frac{9}{2}\right)\right\} = \frac{160}{3}$

(2) $\displaystyle \int_{0}^{1}\left\{\int_{x^2+1}^{x+1}(x^2+2y)dy\right\}dx = \int_{0}^{1}\left[x^2y+y^2\right]_{y=x^2+1}^{y=x+1}dx$

$\displaystyle = \int_{0}^{1}\left[x^2\left\{(x+1)-(x^2+1)\right\}+\left\{(x+1)^2-(x^2+1)^2\right\}\right]dx = \int_{0}^{1}(-2x^4+x^3-x^2+2x)dx$

$\displaystyle = \left[-\frac{2}{5}x^5+\frac{x^4}{4}-\frac{x^3}{3}+x^2\right]_{0}^{1} = -\frac{2}{5}+\frac{1}{4}-\frac{1}{3}+1 = \frac{31}{60}$

(3) $\displaystyle \int_{-2}^{1}\left\{\int_{x^2}^{2-x}(x+y)dy\right\}dx = \int_{-2}^{1}\left[xy+\frac{1}{2}y^2\right]_{y=x^2}^{y=2-x}dx$

$\displaystyle = \int_{-2}^{1}\left\{x(2-x)+\frac{1}{2}(2-x)^2-\left(x^3+\frac{1}{2}x^4\right)\right\}dx = \int_{-2}^{1}\left(-\frac{1}{2}x^4-x^3-\frac{1}{2}x^2+2\right)dx$

$\displaystyle = \left[-\frac{1}{10}x^5-\frac{1}{4}x^4-\frac{1}{6}x^3+2x\right]_{-2}^{1} = \left(-\frac{1}{10}-\frac{1}{4}-\frac{1}{6}+2\right)-\left(\frac{16}{5}-4+\frac{4}{3}-4\right) = \frac{99}{20}$

(4) $\displaystyle \int_{0}^{1}\left(\int_{\sqrt{1-x^2}}^{x+2} xydy\right)dx = \int_{0}^{1}\left[\frac{x}{2}y^2\right]_{y=\sqrt{1-x^2}}^{y=x+2}dx = \int_{0}^{1}\left\{\frac{x}{2}(x+2)^2-\frac{x}{2}(1-x^2)\right\}dx$

$\displaystyle = \int_{0}^{1}\left(x^3+2x^2+\frac{3}{2}x\right)dx = \left[\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{3}{4}x^2\right]_{0}^{1} = \frac{1}{4}+\frac{2}{3}+\frac{3}{4} = \frac{5}{3}$

(5) $\displaystyle \int_{\sqrt{e}}^{e^2}\left(\int_{1}^{x}\frac{1}{xy}dy\right)dx = \int_{\sqrt{e}}^{e^2}\left[\frac{\log y}{x}\right]_{y=1}^{y=x}dx = \int_{\sqrt{e}}^{e^2}\left(\frac{\log x}{x}-\frac{0}{x}\right)dx = \int_{\sqrt{e}}^{e^2}\log x\cdot\frac{1}{x}dx$

$\displaystyle = \left[\frac{1}{2}(\log x)^2\right]_{\sqrt{e}}^{e^2} = \frac{1}{2}\left(4-\frac{1}{4}\right) = \frac{15}{8}$

(6) $\displaystyle \int_{0}^{\frac{\pi}{3}}\left(\int_{0}^{\sin x} xdy\right)dx = \int_{0}^{\frac{\pi}{3}}\left[xy\right]_{y=0}^{y=\sin x}dx = \int_{0}^{\frac{\pi}{3}} x\sin xdx = \left[-x\cos x+\sin x\right]_{0}^{\frac{\pi}{3}}$

$\displaystyle = -\left(\frac{\pi}{3}\cdot\frac{1}{2}-0\cdot 1\right)+\left(\frac{\sqrt{3}}{2}-0\right) = \frac{\sqrt{3}}{2}-\frac{\pi}{6}$

(7) $\displaystyle\int_0^1 \left\{\int_0^x (x+2)e^{2y}dy\right\}dx = \int_0^1 \left[\frac{1}{2}(x+2)e^{2y}\right]_{y=0}^{y=x} dx$

$\displaystyle= \int_0^1 \left\{\frac{1}{2}(x+2)e^{2x} - \frac{1}{2}(x+2)\cdot 1\right\}dx = \left[\frac{1}{4}(x+2)e^{2x} - \frac{1}{8}e^{2x} - \frac{1}{4}(x+2)^2\right]_0^1$

$\displaystyle= \frac{1}{4}(3e^2 - 2\cdot 1) - \frac{1}{8}(e^2-1) - \frac{1}{4}(9-4) = \frac{5}{8}e^2 - \frac{13}{8}$

(8) $\displaystyle\int_0^1 \left(\int_{-x}^{2x} xe^y dy\right)dx = \int_0^1 \left[xe^y\right]_{y=-x}^{y=2x} dx = \int_0^1 (xe^{2x} - xe^{-x})dx$

$\displaystyle= \left[\left(\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right) - (-xe^{-x} - e^{-x})\right]_0^1 = \frac{1}{2}(1\cdot e^2 - 0\cdot 1) - \frac{1}{4}(e^2-1) + (1\cdot e^{-1} - 0\cdot 1) + (e^{-1}-1)$

$\displaystyle= \frac{e^2}{4} + \frac{2}{e} - \frac{3}{4}$

(9) $\displaystyle\int_1^6 \left(\int_1^x \log y dy\right)dx = \int_1^6 \left[y\log y - y\right]_{y=1}^{y=x} dx = \int_1^6 \left\{(x\log x - 1\cdot 0) - (x-1)\right\}dx$

$\displaystyle= \left[\frac{1}{2}x^2\log x - \frac{1}{4}x^2 - \frac{1}{2}(x-1)^2\right]_1^6 = \frac{1}{2}(36\log 6 - 1\cdot 0) - \frac{1}{4}(36-1) - \frac{1}{2}(25-0)$

$\displaystyle= 18\log 6 - \frac{85}{4}$

(10) $\displaystyle\int_1^3 \left(\int_{x^2}^{x^3} \frac{x}{y}dy\right)dx = \int_1^3 \left[x\log y\right]_{y=x^2}^{y=x^3} dx = \int_1^3 x(\log x^3 - \log x^2)dx = \int_1^3 x\log x dx$

$\displaystyle= \left[\frac{1}{2}x^2\log x - \frac{1}{4}x^2\right]_1^3 = \frac{1}{2}(9\log 3 - 1\cdot 0) - \frac{1}{4}(9-1) = \frac{9}{2}\log 3 - 2$

(11) $\displaystyle\int_1^2 \left(\int_1^x \log\frac{x}{y^2}dy\right)dx = \int_1^2 \left\{\int_1^x (\log x - 2\log y)dy\right\}dx$

$\displaystyle= \int_1^2 \left[y\log x - 2(y\log y - y)\right]_{y=1}^{y=x} dx = \int_1^2 \left\{(x-1)\log x - 2(x\log x - 1\cdot 0) + 2(x-1)\right\}dx$

$\displaystyle= \int_1^2 \left\{-x\log x - \log x + 2(x-1)\right\}dx = \left[-\left(\frac{1}{2}x^2\log x - \frac{1}{4}x^2\right) - (x\log x - x) + (x-1)^2\right]_1^2$

$\displaystyle= -\frac{1}{2}(4\log 2 - 1\cdot 0) + \frac{1}{4}(4-1) - (2\log 2 - 1\cdot 0) + (2-1) + (1-0) = \frac{11}{4} - 4\log 2$

(12) $\displaystyle\int_1^2 \left(\int_1^x \frac{x^2}{y}dy\right)dx = \int_1^2 \left[x^2\log y\right]_{y=1}^{y=x} dx = \int_1^2 x^2(\log x - 0)dx = \int_1^2 x^2\log x dx$

$\displaystyle= \left[\frac{1}{3}x^3\log x - \frac{1}{9}x^3\right]_1^2 = \frac{1}{3}(8\log 2 - 1\cdot 0) - \frac{1}{9}(8-1) = \frac{8}{3}\log 2 - \frac{7}{9}$

(13) $\displaystyle\int_0^2 \left(\int_0^{x^2} \frac{x}{1+y}dy\right)dx = \int_0^2 \Big[x\log(1+y)\Big]_{y=0}^{y=x^2}dx = \int_0^2 x\Big\{\log(1+x^2)-0\Big\}dx$

$\displaystyle= \int_0^2 x\log(1+x^2)dx = \left[\frac{1}{2}(x^2+1)\log(1+x^2) - \frac{1}{2}x^2\right]_0^2 = \frac{1}{2}(5\log 5 - 1\cdot 0) - \frac{1}{2}(4-0)$

$\displaystyle= \frac{5}{2}\log 5 - 2$

(14) $\displaystyle\int_1^{\sqrt{3}} \left(\int_x^{x^2} \frac{x}{x^2+y^2}dy\right)dx = \int_1^{\sqrt{3}} \Big[\arctan\frac{y}{x}\Big]_{y=x}^{y=x^2}dx = \int_1^{\sqrt{3}} \Big(\arctan x - \frac{\pi}{4}\Big)dx$

$\displaystyle= \left[x\arctan x - \frac{1}{2}\log(1+x^2) - \frac{\pi}{4}x\right]_1^{\sqrt{3}} = \Big(\sqrt{3}\cdot\frac{\pi}{3} - 1\cdot\frac{\pi}{4}\Big) - \frac{1}{2}(\log 4 - \log 2) - \frac{\pi}{4}(\sqrt{3}-1)$

$\displaystyle= \frac{\sqrt{3}}{12}\pi - \frac{1}{2}\log 2$